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Abstract: We derive, using the pure-spinor formalism, the complete – including the fermions – four-point effective action of both type II superstrings to all orders in α', at tree level in string loops. We find that, in the quartic-field approximation, the supergravity Lagrangian can be thought of as the tensor product, in a suitable sense, of two copies of the superYang-Mills Lagrangian in ten dimensions. The NS-NS three-form enters the supergravity Lagrangian through a modified connection with torsion. As a byproduct, we derive the complete, i.e. to all orders in the θ-expansion, closed-string vertex operator in a flat target-space background.

Keywords: Superstrings, Supergravity, Pure spinors.

Contents

1. Introduction

The pure-spinor formalism for the superstring introduced a few years ago in [1] (see [2] for a review), has been a remarkable technical advance, overcoming long-standing difficulties with both the Green-Schwarz and the RNS formalisms. Pure spinors were originally introduced in ten-dimensional superYang-Mills in an attempt to construct an off-shell formulation [3]. It was subsequently noted that the on-shell constraints in both ten- and eleven-dimensional supersymmetric theories [4, 5] can be thought of as pure-spinor integrability conditions. Eventually Berkovits incorporated these early insights into a full-fledged formulation of string theory. Berkovits' proposal is self-consistent and unambiguous, and has already passed a number of nontrivial tests [6, 7, 8, 9, 10, 11, 12]. However the origin of some of the prescriptions of the pure-spinor formalism remains largely mysterious. Perhaps most significantly, it has proven impossible so far to understand the pure-spinor BRST cohomology operator as coming from the gauge-fixing of a reparameterization-invariant action [1]. As a consequence, the computation of scattering amplitudes does not follow from a path integral quantization à la Polyakov, but has to be performed using a set of more-or-less ad-hoc rules.

There have been several attempts in the literature to justify these rules and 'derive' the formalism from different points of view. An alternative formulation which dispenses with using pure spinors by embedding Berkovits' theory in a larger theory with additional fields, was put forward in [14, 15]. This model can then be understood formally as a WZW model with $\mathcal{N} = 2$ supersymmetry based on a superalgebra which is a fermionic central extension of the Poincaré superalgebra of the target space [16]. One can recover the correct spectrum in a completely covariant manner [17]. Recently it has been shown that the $\mathcal{N} = 2$ algebra can also arise from a different set of non-minimal fields [18]. For other approaches, see also [13], [19], [20].

Although it is quite important to put the pure-spinor formalism on a sound conceptual footing, there is by now little doubt that it works, and for practical computations it is likely to be the most economical one. For this reason, in the present paper we adopt a pragmatic approach, and simply take advantage of the natural, Poincaré-covariant way in which the fermions and the Ramond-Ramond fields are described in the pure-spinor formalism, in order to derive the complete – fermions included – four-point tree-level effective action of both type II superstrings to all orders in α'. Our approach makes transparent a simple relation, which was already known to hold in the NS-NS sector [21, 22, 23], between the four-point gravitational action and the superYang-Mills action in ten dimensions. I.e. the former is the tensor product (in a sense which we make precise in section 5.2) of two copies of the latter,

$$\mathcal{L}_{SUGRA} = \mathcal{L}_{SYM} \otimes \widetilde{\mathcal{L}}_{SYM} \ . \tag{1.1}$$

We stress that this only holds at the level of the four-field approximation. Equation (1.1) is a direct consequence of the tree-level relations between open- and closed-string amplitudes, first observed in [24]. We give a general proof of (1.1), using the Kawai-Lewellen-Tye relations, in section 5.2. We have

[1] In [13] it was shown that the pure-spinor formalism can be derived by gauge-fixing a reparameterization-invariant action of Green-Schwarz type. So far, however, this procedure has not proven useful in understanding the origin of Berkovits' prescription for the scattering amplitudes, since it runs into the well-known problems associated with the semi-light-cone gauge.

also verified it explicitly by deriving the four-point Lagrangian in both ways: by direct computation of the closed-string four-point amplitudes, as well as by 'squaring' the SYM Lagrangian.

The consequences of the KLT relations have been investigated by Z. Bern and collaborators (see for instance [25] and references therein), who have found a way of considerably simplifying perturbative computations in pure gravity by systematically exploiting the factorization property and other inputs from string theory. In the present paper we focus on the quartic part of the full supergravity Lagrangian; we do not make any attempt to generalize the constructions of Bern et al to the complete type II supergravity – although that would certainly be interesting.

In hindsight the present paper could have been written shortly after the publication of [24]: anticipating the advance of a covariant formulation of the superstring, one could have postulated a manifestly Poincaré-invariant open-string four-point amplitude. This amplitude would be the covariantization of the four-point RNS or Green-Schwarz amplitude, and would be such that it reproduces the known four-field, four-derivative correction to the superYang-Mills action in ten dimensions. Indeed, as we shall see in section 4, the open-string amplitude derived using the pure-spinor approach has precisely these properties. The closed-string Lagrangian would then follow form the tensor product (1.1).

As a by-product of the present investigation, we are able to derive the complete, i.e. to all orders in the θ-expansion, closed-string vertex operator in a flat target-space background. The θ-expansion of the closed-string vertex operators for type II superstrings in curved backgrounds was considered in [26], where an iterative algorithm for computing the expansion at each order in θ was presented. The procedure of [26] is equivalent to the normal-coordinate expansion in superspace [27], which was first developed in the context of four-dimensional $\mathcal{N} = 1$ supergravity. The normal coordinate expansion has recently been applied by one of the present authors to the case of eleven-dimensional supergravity [28]. As was observed in that reference, in the linear approximation the θ-expansion can be explicitly computed to all orders. This fact was then used in [28] to derive the membrane vertex operator in flat target space, to all orders in θ. As we show in section 3, an all-order result is also possible in the case of the superstring.

There already exists a large amount of literature on higher-order α'-corrections to ordinary supergravities in ten dimensions, see [30] for a review and a more extensive list of references. In type II theories most of our information about these corrections comes from perturbative string-theory computations. However, at the level of the four-field approximation, the eight-derivative correction to the type IIB supergravity is implicitly known in superspace: it is captured by the chiral integral (i.e. the integral over half the superspace) of the fourth power of the linearized scalar superfield of [31]. Unfortunately, the complete component form of the action is somewhat tedious to extract, although it is known to reproduce the $t_8 t_8 R^4$ part of the Lagrangian [32]. An interesting diagrammatic technique to perform the superspace integral has recently been proposed in [33]. It seems unlikely to us, however, that the computation in the present paper would be made any easier by adopting the methods of [33]. We comment further on the linearized superfield in section 6.

The RR sector of type IIB superstrings was considered by Peeters and Westerberg in [34]. These authors were able to perform the formidable task of obtaining the Poincaré-covariant form of the two-graviton, two-RR field scattering amplitudes, working at first order in string loops within the RNS formalism. The methods of [34] in handling the RR sector, which make use of earlier techniques

developed some time ago by Atick and Sen [35], appear rather convoluted and tedious compared to the pure-spinor approach adopted in the present paper. As we shall see in section 5.3, our results are in agreement with those of [34].

Without further ado, let us present here the main result of our paper: to all orders in α', in the four-field approximation, the effective Lagrangian reads

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2\kappa^2} R - \frac{1}{2}\partial_m D \partial^m D - \frac{1}{3!} e^{-\sqrt{2}\kappa D} H_{mnp} H^{mnp} - \frac{1}{\sqrt{2}\kappa} \sum_M \frac{1}{M!} e^{\frac{5-M}{\sqrt{2}}\kappa D} \widehat{F}_{m_1\ldots m_M} \widehat{F}^{m_1\ldots m_M} \\
+ \sqrt{2} \sum_{M+N=8} (-1)^{[\frac{M+1}{2}]} \star (B \wedge F_{(M)} \wedge F_{(N)}) \\
+ \quad \widehat{\mathcal{G}}(\partial, \alpha') \Big\{ \frac{1}{4!} t^{a_1\ldots a_8} t_{b_1\ldots b_8} \widehat{R}_{a_1a_2}{}^{b_1b_2} \widehat{R}_{a_3a_4}{}^{b_3b_4} \widehat{R}_{a_5a_6}{}^{b_5b_6} \widehat{R}_{a_7a_8}{}^{b_7b_8} \\
+ \sum_{M,N} u_{ij}{}^{mnpqm'n'p'q';a_1\ldots a_M;b_1\ldots b_N} \widehat{R}_{mnm'n'} \widehat{R}_{pqp'q'} \partial^i F_{a_1\ldots a_M} \partial^j F_{b_1\ldots b_N} \\
+ \sum_{M,N,P,Q} v^{a_1\ldots a_M;b_1\ldots b_N;c_1\ldots c_P;d_1\ldots d_Q} \partial_i\partial_j F_{a_1\ldots a_M} \partial^i\partial^j F_{b_1\ldots b_N} F_{c_1\ldots c_P} F_{d_1\ldots d_Q} \Big\} + \mathcal{O}(\psi^2) \ ,
\end{aligned}
\tag{1.2}
$$

where κ is the gravitational coupling constant, and the action of the operator $\widehat{\mathcal{G}}$ is described below (5.36). The omitted fermionic terms are given explicitly in (5.36). The tensors u, v, are defined in appendix E; our normalization for t_8 can be found in section 5.2. The sums over $M, \ldots Q$, run over even integers from zero to four for IIA supergravity, and over odd integers from one to five for IIB. In the case of IIB we work in a formalism with an action, so that the self-duality of the five-form (at lowest order in α') is imposed at the level of the equations of motion. The Riemann tensor $\widehat{R}$ is with respect to a modified connection with torsion, which includes the NS-NS three-form field and the dilaton, see equation (5.18) below. In fact, the complete Lagrangian could be written in terms of $\widehat{R}$ instead R, as the scalar curvature of $\widehat{R}$ differs from that of R by a term proportional to $\nabla^2 D$, which is a total derivative. The RR field strengths $\widehat{F}$ appearing in the Lagrangian above obey modified Bianchi identities and are explicitly defined in (5.11).

The structure of this paper is as follows. In section 2 we give a short reminder of the ingredients of the pure-spinor string which are used in the present paper. In section 3 we give the all-order expansion of the vertex operators in flat target space. In section 4 we compute the open-string 4-point amplitude and the corresponding effective action. In section 5 we 'square' the open-string amplitudes, according to the KLT relations, and construct the complete IIA/IIB effective action at quartic order in the fields. In section 6 we discuss our results in relation to the predictions of the linearized superfield. Finally, in section 7, we comment on possible extensions and applications of our work. The appendices contain further technical details of the calculation.

2. Review of the pure-spinor formalism

We now give a brief review of the pure-spinor formalism for the superstring, focusing on the points relevant to the present paper. The material in this section is well-known; it is included here mainly

for the purpose of establishing notation/conventions. In the following we are adopting conventions as in e.g. [9], restoring α' in addition.

The pure-spinor string is based on a worldsheet conformal field theory with fields x^m, θ^α corresponding to the coordinates of the target superspace, and a (worldsheet scalar, target-space spinor) ghost λ^α, plus the conjugate momenta. For the type IIA (resp. IIB) string, we also have the right-movers $\tilde{\theta}^{\bar{\alpha}}$, $\tilde{\lambda}^{\bar{\alpha}}$, with chirality opposite to (resp. same as) that of their left-moving counterparts. The worldsheet action in a flat background is given by

$$S = \int d^2z\{-\frac{1}{2}\partial x^m \bar{\partial} x_m - p_\alpha \bar{\partial}\theta^\alpha - \tilde{p}_{\bar{\alpha}}\partial\tilde{\theta}^{\bar{\alpha}} + w_\alpha \bar{\partial}\lambda^\alpha + \tilde{w}_{\bar{\alpha}}\partial\tilde{\lambda}^{\bar{\alpha}}\}\,. \tag{2.1}$$

This looks like a free action, but the ghosts are not really free fields as they are subject to the pure-spinor constraint

$$\lambda\gamma^m\lambda = 0\,. \tag{2.2}$$

This constraint has several consequences: first, it reduces the number of independent components of λ from 32 to 22, so that the central charge vanishes and the theory is critical in ten dimensions. The stress energy tensor is

$$\begin{aligned} T &= -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + w_\alpha \partial\lambda^\alpha \\ \tilde{T} &= -\frac{1}{2}\bar{\partial} x^m \bar{\partial} x_m - \tilde{p}_{\bar{\alpha}}\bar{\partial}\tilde{\theta}^{\bar{\alpha}} + \tilde{w}_{\bar{\alpha}}\bar{\partial}\tilde{\lambda}^{\bar{\alpha}}\ . \end{aligned} \tag{2.3}$$

Secondly, the constraint implies the presence of a gauge symmetry for the momentum:

$$\delta w_\alpha = \Lambda^m \gamma_m \lambda\,. \tag{2.4}$$

The observables of the theory must be gauge-invariant, and one can show that they can always be expressed in terms of the currents

$$\begin{aligned} J &:= (w\lambda) \\ N_{mn} &:= \frac{1}{2}(w\gamma_{mn}\lambda)\ , \end{aligned} \tag{2.5}$$

which are respectively the ghost-number current and the Lorentz generators in the ghost sector. As it turns out, one can write down a BRST operator to select the physical states. Let us first make the following definitions:

$$\begin{aligned} d_\alpha &:= p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial x_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta) \\ \Pi^m &:= \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)\ . \end{aligned} \tag{2.6}$$

From the action one can derive the following OPE's

$$\begin{aligned} x^m(y)x^n(z) &\sim -\alpha'\eta^{mn} log\frac{|y-z|^2}{\alpha'}\,, & p_\alpha(y)\theta^\beta(z) &\sim \frac{\alpha'}{y-z}\delta_\alpha^\beta \\ d_\alpha(y)d_\beta(z) &\sim -\frac{\alpha'}{y-z}\gamma^m_{\alpha\beta}\Pi_m\,, & d_\alpha(y)\Pi^m(z) &\sim \frac{\alpha'}{y-z}(\gamma^m\partial\theta)_\alpha \\ d_\alpha(y)\partial\theta^\beta(z) &\sim \frac{\alpha'}{(y-z)^2}\delta_\alpha^\beta\,, & \Pi^m(y)\Pi^n(z) &\sim -\frac{\alpha'}{(y-z)^2}\eta^{mn}\ . \end{aligned} \tag{2.7}$$

For any superfield $V(x,\theta)$,

$$\Pi^m(y)V(z) \sim -\frac{\alpha'}{y-z}\partial^m V\,,$$
$$d_\alpha(y)V(z) \sim \frac{\alpha'}{y-z}D_\alpha V\ , \tag{2.8}$$

where $D_\alpha := \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ is the superderivative. The BRST operator is

$$Q := \oint dz \lambda^\alpha d_\alpha\ . \tag{2.9}$$

It can easily be checked that $Q^2 = 0$, using the OPE's given above and the constraint (2.2). The physical states are determined by the cohomology of Q. For massless states, one writes down the most general form for the vertex operators

$$U := \lambda^\alpha A_\alpha(x,\theta)$$
$$V := \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2}N^{mn}F_{mn} \tag{2.10}$$

and requires that they satisfy $QU = 0$, $QV = \alpha'\partial U$, so that $\int V$ is BRST-closed. In the next section we will find the solutions of these conditions. When studying closed strings one has to consider also the right-moving sector. The total BRST operator is then $Q + \tilde{Q}$, and the vertex has the form $U_{closed} = \lambda^\alpha\tilde{\lambda}^{\bar{\alpha}}A_{\alpha\bar{\alpha}}$. Furthermore, one can take advantage of the fact that the left- and right-movers are decoupled, and the cohomology of the total BRST is simply the tensor product of the two sectors. This means that every closed vertex is a linear combination of vertices of a factorized form:

$$U_{closed} := e^{ik\cdot x}\lambda^\alpha A_\alpha(\theta)\tilde{\lambda}^{\bar{\beta}}\tilde{A}_{\bar{\beta}}(\bar{\theta})$$
$$V_{closed} := e^{ik\cdot x}(\partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2}N^{mn}F_{mn})$$
$$\otimes\,(\bar{\partial}\tilde{\theta}^{\bar{\beta}}\tilde{A}_{\bar{\beta}} + \tilde{\Pi}^m\tilde{A}_m + \tilde{d}_{\bar{\beta}}\tilde{W}^{\bar{\beta}} + \frac{1}{2}\tilde{N}^{mn}\tilde{F}_{mn})\ . \tag{2.11}$$

Note that the factorization takes place for each given momentum k, which must take the same value in both sectors.

Tree level amplitudes

The N-point tree level scattering amplitude is given by (concentrating on the left-movers)

$$\mathcal{A} = \langle U_1(z_1)U_2(z_2)U_3(z_3)\int dz_4 V_4(z_4)\ldots\int dz_N V_N(z_N)\rangle\ . \tag{2.12}$$

Let us assume that after integrating out all nonzero modes the amplitude (2.12) takes the form

$$\mathcal{A} = \int dz_4\ldots\int dz_N\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r,k_r,\theta)\rangle\ , \tag{2.13}$$

where k_r^m, $r = 1\ldots N$ are the momenta and $f_{\alpha\beta\gamma}$ depends only on the zero modes of θ^α. The prescription for integrating over the zero modes is

$$\mathcal{A} = T^{\alpha\beta\gamma}_{\rho\sigma\tau}(\frac{\partial}{\partial\theta}\gamma^{pmn}\frac{\partial}{\partial\theta})(\gamma_p\frac{\partial}{\partial\theta})^\rho(\gamma_m\frac{\partial}{\partial\theta})^\sigma(\gamma_n\frac{\partial}{\partial\theta})^\tau\int dz_4\ldots\int dz_N f_{\alpha\beta\gamma}(z_r,k_r,\theta)\ , \tag{2.14}$$

where

$$T^{\alpha\beta\gamma}_{\rho\sigma\tau} := \frac{1}{672}(\delta^{(\alpha}_{\rho}\delta^{\beta}_{\sigma}\delta^{\gamma)}_{\tau} - \frac{3}{20}\gamma^{(\alpha\beta}_{m}\delta^{\gamma)}_{(\rho}\gamma^{m}_{\sigma\tau)}) \ . \tag{2.15}$$

The tensor $T^{\alpha\beta\gamma}_{\rho\sigma\tau}$ satisfies $T^{\alpha\beta\gamma}_{\rho\sigma\tau}\gamma^{m}_{\alpha\beta} = T^{\alpha\beta\gamma}_{\rho\sigma\tau}\gamma^{\rho\sigma}_{m} = 0$ and has been normalized so that $T^{\alpha\beta\gamma}_{\alpha\beta\gamma} = 1$.

3. θ-expansions and the vertex operator

In this section we give the details relating to the θ-expansion of the closed-string vertex operators for type II superstrings in a flat-space background. As was already noted in the introduction, we will show that it is possible to obtain an explicit exact result to all orders in θ. Since, as we have seen in the previous section, the closed-string vertex operators factorize in flat target-space, cf. eqn (2.11), it suffices to consider the open-string vertex operator.

Ten-dimensional superYang-Mills admits a formulation in superspace in terms of on-shell superfields. Let F be the supercurvature two-form corresponding to superpotential one-form A, $F = DA$. It has been known for some time that imposing the constraint $F_{\alpha\beta} = 0$ on the spinor-spinor components of the supercurvature, leads to the superYang-Mills equations of motion [3]. The physical multiplet consists of the gauge boson and the gaugino, which can be identified with the $\theta = 0$ components of the superfields A_m, W^α, respectively. On-shell these superfields satisfy

$$\begin{aligned} 2D_{(\alpha}A_{\beta)} &= \gamma^{m}_{\alpha\beta}A_m \\ D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn} \ , \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} F_{mn} &= 2\partial_{[m}A_{n]} \\ W^\alpha &= \frac{1}{10}(\gamma^m)^{\alpha\beta}(D_\beta A_m - \partial_m A_\beta) \end{aligned} \tag{3.2}$$

and D_α is the supercovariant spinor derivative. Imposing the gauge (this is the analogue of the choice of normal coordinates in the case at hand):

$$\theta^\alpha A_\alpha = 0 \ , \tag{3.3}$$

allows us to convert the supercovariant spinor derivative into an ordinary one: $\theta^\alpha D_\alpha = \theta^\alpha \partial/\partial\theta^\alpha$. Taking (3.1) into account, this leads immediately to the following recursion relations:

$$\begin{aligned} A^{(n)}_\alpha &= \frac{1}{n+1}(\gamma^m\theta)_\alpha A^{(n-1)}_m \\ A^{(n)}_m &= \frac{1}{n}(\theta\gamma_m W^{(n-1)}) \\ W^{\alpha(n)} &= -\frac{1}{2n}(\gamma^{mn}\theta)^\alpha \partial_m A^{(n-1)}_n \ . \end{aligned} \tag{3.4}$$

These can be solved to give

$$\begin{aligned} A^{(2k)}_m &= \frac{1}{(2k)!}[\mathcal{O}^k]_m{}^q a_q \\ A^{(2k+1)}_m &= \frac{1}{(2k+1)!}[\mathcal{O}^k]_m{}^q(\theta\gamma_q\xi) \ , \end{aligned} \tag{3.5}$$

where

$$[\mathcal{O}]_m{}^q := \frac{1}{2}(\theta\gamma_m{}^{qp}\theta)\partial_p \tag{3.6}$$

and we have set

$$a_m := A_m|; \quad \xi^\alpha := W^\alpha|; \quad f_{mn} := F_{mn}| \ . \tag{3.7}$$

In the equation above, we use the standard notation according to which $S|$ denotes the $\theta = 0$ component of the superfield S. Clearly, (3.4,3.5) completely determine the θ-expansions of all superfields. The first few terms in the expansions read:

$$\begin{aligned} A^{(1)}_\alpha = \frac{1}{2}(\theta\gamma^m)_\alpha a_m\,, \quad A^{(2)}_\alpha = \frac{1}{3}(\theta\gamma^m)_\alpha(\theta\gamma_m\xi)\,, \quad A^{(3)}_\alpha = \frac{1}{16}(\theta\gamma^m)_\alpha(\theta\gamma_m{}^{pq}\theta)\partial_q a_p\ ; \\ A^{(1)}_m = (\theta\gamma_m\xi)\,, \quad A^{(2)}_m = \frac{1}{4}(\theta\gamma_m{}^{pq}\theta)\partial_q a_p\,, \quad A^{(3)}_m = \frac{1}{12}(\theta\gamma_m{}^{qp}\theta)(\theta\gamma_q\partial_p\xi)\ . \end{aligned} \tag{3.8}$$

The series in (3.5) can be formally summed to all orders[2] in θ to give

$$A_m = [\cosh\sqrt{\mathcal{O}}]_m{}^q a_q + [\mathcal{O}^{-1/2}\sinh\sqrt{\mathcal{O}}]_m{}^q(\theta\gamma_q\xi)\ . \tag{3.9}$$

It is interesting to observe the remarkable similarity of the expression above to the θ-expansion of the gravitino field strength in eleven-dimensional supergravity, equation (106) of [28]. Similar expressions can readily be obtained for A_α, W^α.

These formulæ also give the complete expansion of the closed string vertex, when the polarizations of the 'graviton' Θ_{mn} (which here includes the dilaton and the antisymmetric tensor), the gravitini ψ^α_m, $\bar\psi^{\bar\alpha}_m$, and the RR bispinor field-strength $\not{F}^{\alpha\bar\beta}$ are also assumed to be factorized,

$$\Theta_{mn} := a_m \otimes \tilde a_n; \quad \bar\psi^{\bar\alpha}_m := i\sqrt{2}a_m \otimes \tilde\xi^{\bar\alpha}; \quad \psi^\alpha_m := i\sqrt{2}\xi^\alpha \otimes \tilde a_m; \quad \not{F}^{\alpha\bar\beta} := \sqrt{\kappa}\xi^\alpha \otimes \tilde\xi^{\bar\beta}\ , \tag{3.10}$$

where κ is the gravitational coupling constant (normalizations have been chosen for later convenience). In conclusion: plugging the expressions obtained here for the θ-expansions of the various superfields in (2.10, 2.11), we obtain the explicit form of the string vertices in flat target space to all orders in θ, as advertised.

4. The open-string amplitude

We now have all the ingredients to compute the amplitudes. In order to improve the presentation, in the following we will confine ourselves to presenting the main results of the computation. The interested reader may consult the appendices for the omitted technical details. The first step is to compute the open-string amplitudes, i.e. we consider only the left-moving sector. The x-dependent part of the correlator is standard:

$$\begin{aligned} \langle e^{ik_1\cdot x(z_1)}e^{ik_2\cdot x(z_2)}e^{ik_3\cdot x(z_3)}e^{ik_4\cdot x(z_4)}\rangle = \prod_{i<j}^4 (z_i - z_j)^{\alpha' k_i\cdot k_j} \equiv \Pi(z_{ij})\,, \\ \langle e^{ik_1\cdot x(z_1)}e^{ik_2\cdot x(z_2)}e^{ik_3\cdot x(z_3)} : \partial x^m(z_4)e^{ik_4\cdot x(z_4)} :\rangle = \sum_{i=1}^3 \frac{i\alpha' k^m_i}{z_i - z_4}\Pi(z_{ij})\ . \end{aligned} \tag{4.1}$$

[2]Of course the series terminates at θ^{16}.

4.1 The 3-point amplitude

The computation of the 3-point amplitude is straightforward, since it only contains unintegrated vertices. We have

$$\mathcal{A}_3 = \langle U_1(z_1)U_2(z_2)U_3(z_3)\rangle \ . \tag{4.2}$$

For three massless particles on-shell, $k_i \cdot k_j = 0$. We find

$$\mathcal{A}_3^{\rm op}(k_i; a_i, \xi_i) = \frac{1}{5760}\Big[k_1 \cdot a_3\, a_1 \cdot a_2 + k_2 \cdot a_1\, a_2 \cdot a_3 + k_3 \cdot a_2\, a_1 \cdot a_3 \\ - i\xi_2 \not{a}_1 \xi_3 + i\xi_1 \not{a}_2 \xi_3 - i\xi_1 \not{a}_3 \xi_2\Big] \, . \tag{4.3}$$

4.2 The 4-point amplitude

Here we compute

$$\mathcal{A}_4 = \langle U_1(z_1)U_2(z_2)U_3(z_3)\int d^2 z_4 V_4(z_4)\rangle \ . \tag{4.4}$$

This amplitude receives several contributions, coming from the different terms in V_4 and from the expansion of all the fields in powers of θ. For each of these terms, the function of the zero modes $f_{\alpha\beta\gamma}$ in (2.13) has a different structure. The detailed computation of all the terms in the 4-point amplitude can be found in appendix C. We give here the final result:

$$\frac{\alpha'}{5760}\Big[\frac{\Pi(z_{ij})}{z_1 - z_4}\Big\{\ 2k_1 \cdot a_4\, k_2 \cdot a_3\, a_1 \cdot a_2 + 2k_3 \cdot a_4\, k_2 \cdot a_1\, a_2 \cdot a_3 + 2k_1 \cdot a_3\, k_3 \cdot a_2\, a_1 \cdot a_4 \\ - 2k_4 \cdot a_1\, k_2 \cdot a_3\, a_2 \cdot a_4 + k_1 \cdot k_4\, a_1 \cdot a_2\, a_3 \cdot a_4 - k_2 \cdot k_4\, a_2 \cdot a_3\, a_1 \cdot a_4 \\ + 2i\xi_1 \not{a}_2 \xi_4\, k_2 \cdot a_3 \ + 2i\xi_2 \not{a}_3 \xi_4\, k_4 \cdot a_1 \ + i\xi_1 \not{k}_3 \xi_4 a_2 \cdot a_3 + i\xi_3 \not{a}_2 \not{a}_1 \not{k}_1 \xi_4 \\ + i\xi_2 \not{a}_1 \xi_3\, k_1 \cdot a_4 \ - i\xi_2 \not{a}_4 \xi_3\, k_4 \cdot a_1 \ + 2i\xi_1 \not{a}_3 \xi_2\, k_1 \cdot a_4 \ + i\xi_2 \not{k}_4 \xi_3 a_1 \cdot a_4 \\ - i\xi_3 \not{a}_2 \not{a}_4 \not{k}_4 \xi_1 - (\xi_1 \gamma^m \xi_4)(\xi_2 \gamma_m \xi_3)\Big\}\Big] + \text{cyclic permutations.}$$

After doing the z_4 integration (see appendix D) and then summing $(2 \leftrightarrow 3)$ we get

$$\mathcal{A}_4 = \frac{2\alpha'^2}{5760}\Big\{ - 2k_1 \cdot k_3\, k_1 \cdot a_4\, k_2 \cdot a_3\, a_1 \cdot a_2 - \frac{1}{2} k_1 \cdot k_4\, k_2 \cdot k_4\, a_1 \cdot a_2\, a_3 \cdot a_4 \\ - ik_1 \cdot k_3 \Big[a_2 \cdot a_3\, (\xi_1 \not{k}_3 \xi_4) - k_3 \cdot a_2\, (\xi_1 \not{a}_3 \xi_4) + k_2 \cdot a_3\, (\xi_1 \not{a}_2 \xi_4)\Big] \\ + ik_1 \cdot k_4 \Big[k_1 \cdot a_3\, (\xi_1 \not{a}_2 \xi_4) + \frac{1}{2}(\xi_1 \not{a}_3 \not{k}_3 \not{a}_2 \xi_4)\Big] \\ - \frac{1}{3} k_1 \cdot k_2\ (\xi_1 \gamma_m \xi_4)(\xi_2 \gamma^m \xi_3)\Big\} + \text{permutations} \, . \tag{4.5}$$

4.3 The open-string effective action

The action which reproduces the amplitude (4.5) is

$$S_2 \propto \alpha'^2 \int d^{10}x\, \mathcal{G}(\partial, \alpha')\Big\{ \text{tr} f^4 - \frac{1}{4}(\text{tr} f^2)^2 - 4i f_{mn} f_{mp} (\xi \gamma^n \partial^p \xi) - 2i f_{mn} f_{pq} (\xi \gamma^{mnp} \partial^q \xi) \\ + \frac{4}{3} (\xi \gamma_m \partial_n \xi)(\xi \gamma^m \partial^n \xi)\Big\} \ , \tag{4.6}$$

where the operator $\mathcal{G}(\partial, \alpha')$ should be understood as follows: one has to split the positions of the different fields, take the Fourier transform and insert $\mathcal{G}(k_i)$, which is defined in appendix D.

5. The closed-string amplitude

The closed-string amplitude can be readily reconstructed using the formulæ of [24]. In particular, denoting by $\mathcal{A}_N^{op}$ the N-point open string amplitude, the N-point closed string amplitude reads:

$$\mathcal{A}_N^{\rm cl} =(\frac{i}{2})^{N-3}g^{N-2}\sum_{P,P'}\mathcal{A}_N^{\rm op}(P)\otimes\tilde{\mathcal{A}}_N^{\rm op}(P')e^{i\pi f(P,P')}\,. \tag{5.1}$$

Here the sum is over different orderings P, P' of the open string vertices. In the case of 3 and 4 points the sum actually consists of only one term. We will need the explicit expression for these cases:

$$\mathcal{A}_3^{\rm cl} =g\,\mathcal{A}_3^{\rm op}\otimes\tilde{\mathcal{A}}_3^{\rm op} \tag{5.2}$$

and

$$\mathcal{A}_4^{\rm cl} = -\,g^2\sin(\pi\alpha' k_2\cdot k_3)\mathcal{A}_4^{\rm op}(\alpha' s/2,\alpha' t/2)\otimes\tilde{\mathcal{A}}_4^{\rm op}(\alpha' t/2,\alpha' u/2)\,. \tag{5.3}$$

Here s, t, u are the usual Mandelstam variables for 4-particle scattering:

$$s=-2k_{12}=-2k_{34}\,;t=-2k_{14}=-2k_{23}\,;u=-2k_{13}=-2k_{24}\,.$$

The constant g in (5.2),(5.3) is the closed string coupling, and the amplitudes on the right hand side should be taken without the corresponding open string coupling. It follows in particular that the normalization of the open amplitudes is irrelevant for this computation, since it can be changed by redefining the open string coupling.

5.1 The 3-point amplitude

At the level of 3-point amplitudes, the normalization can be determined by matching with the coefficients of the quadratic effective action. For higher-point functions, the requirement of unitarity is sufficient to fix the normalization; for instance, the 4-point function will have poles that must come from 1-particle-reducible diagrams, so their coefficient is determined by the 3-point amplitudes. All this is well-known and for the purely-gravitational part of the effective action the computations have been explained in detail by Gross and Sloan [21]. We follow their conventions and normalizations in this section.

Using (4.3) and (5.2) we find for the bosonic part:

$$\begin{aligned}\mathcal{A}_{3b}^{\rm cl} = g\Big(&k_2^m k_2^n\Theta_{1,mn}\Theta_{2,pq}\Theta_3^{pq}+k_3^m k_2^q\Theta_{2,mn}\Theta_{3,pn}\Theta_{1,pq}+k_1^m k_2^q\Theta_{3,mn}\Theta_{2,pn}\Theta_{1,pq}\\ &+\frac{1}{\kappa}\not{F}_1^{\alpha\bar\alpha}\not{F}_2^{\beta\bar\beta}\Theta_{3,mn}\gamma^m_{\alpha\beta}\gamma^n_{\bar\alpha\bar\beta}+\text{cyclic}\Big)\,,\end{aligned} \tag{5.4}$$

where $\Theta_{mn}:=a_m\otimes\tilde a_n$, $\not{F}^{\alpha\bar\alpha}:=\sqrt{\kappa}\,\xi^\alpha\otimes\tilde\xi^{\bar\alpha}$, and $g=2\kappa$. Moreover in our conventions the symmetric, traceless part of Θ_{mn} is given by h_{mn}, $g_{mn}=\eta_{mn}+2\kappa h_{mn}$; the antisymmetric part is B_{mn}, $H_{mnp}=3\partial_{[m}B_{np]}$; the trace part is set equal to $(\eta_{mn}-k_m\tilde k_n-\tilde k_m k_n)D/\sqrt{8}$.

The fermionic part of the amplitude can be written as

$$\begin{aligned}\mathcal{A}_{3f}^{\rm cl} = -\frac{g}{2}\Big(&-ik_1^{[m}h_1^{n]p}\psi_{2m}\gamma_p\psi_{3n}-\frac{i}{2}k_1^n h_{3np}\psi_{1m}\gamma^p\psi_2^m+(\psi\to\bar\psi)\\ &-\frac{1}{\sqrt{\kappa}}\psi_{2m}\gamma^n\not{F}_3\gamma^m\bar\psi_{1n}+\text{permutations}\Big)\,.\end{aligned} \tag{5.5}$$

Fermionic terms

The fermionic field $\bar{\psi}_m^{\bar{\alpha}} := i\sqrt{2}a_m \otimes \tilde{\xi}^{\bar{\alpha}}$, can be decomposed on-shell into the gamma-traceless (spin 3/2) and gamma (spin 1/2) parts: $\psi_m = \chi_m + \frac{i}{\sqrt{8}}\gamma_m\lambda$. In terms of the χ, λ fields, the fermionic kinetic terms read

$$\frac{1}{2}\chi_m\gamma^{mnp}D_n\chi_p + \frac{1}{2}\lambda\gamma^m D_m\lambda + (\chi \to \bar{\chi}, \lambda \to \bar{\lambda})\,. \tag{5.6}$$

In this equation, $D_m\chi_n = \partial_m\chi_n - \frac{1}{4}\omega_m{}^{ab}\gamma_{ab}\chi_n$, and the linearized spin connection is given by $\omega_{mab} = 2\kappa\partial_{[a}h_{b]m}$. On the other hand, the fermionic 3-point amplitude (5.5) corresponds to the Lagrangian

$$\begin{aligned}&-\kappa\big(\frac{1}{2}\Theta_{np}\psi_m\gamma^p\partial^n\psi^m + \partial_m\Theta_{np}\psi^m\gamma^p\psi^n\big) + (\psi \to \bar{\psi})\\ &+\sqrt{\kappa}\psi_m\gamma^p \not{F}\gamma^m\bar{\psi}_p\,.\end{aligned} \tag{5.7}$$

When Θ is the graviton, it can be seen that these couplings are accounted for by the three-point contribution coming from the kinetic term (5.6). The remaining contribution (obtained by letting Θ be the dilaton or the antisymmetric tensor) cannot come from the kinetic term. We thereby obtain the fermionic 3-point Lagrangian (completed as usual by the appropriate dilaton couplings)

$$\begin{aligned}&\frac{\kappa}{24}e^{-\frac{\kappa D}{\sqrt{2}}}H^{m_1m_2m_3}(\chi_n\gamma^{[n}\gamma_{m_1m_2m_3}\gamma^{p]}\chi_p) + (\chi \to \bar{\chi})\\ &-2\sqrt{\kappa}\sum_p \frac{c_p}{p!}e^{\frac{5-p}{2\sqrt{2}}\kappa D}F^{m_1\dots m_p}(\chi_n\gamma^{[n}\gamma_{m_1\dots m_p}\gamma^{p]}\bar{\chi}_p) + \dots\ ,\end{aligned} \tag{5.8}$$

where we have expanded

$$\not{F}^{\alpha\bar{\alpha}} = \sum_p \frac{c_p}{p!}(\gamma_{m_1\dots m_p})^{\alpha\bar{\alpha}}F^{m_1\dots m_p}; \qquad c_p^2 = \frac{(-1)^{p+1}}{16\sqrt{2}}\ . \tag{5.9}$$

The ellipses in equation (5.8) signify $\mathcal{O}(\lambda^2)$, as well as $\mathcal{O}(\chi\lambda)$ terms, which we have omitted for simplicity. The interested reader can find the complete fermionic Lagrangian in the literature [3]. In order to bring the three-point couplings in the form above, we added to (5.7) certain terms which vanish on-shell.

Bosonic terms

The bosonic amplitude (5.4) corresponds to an effective 3-point Lagrangian

$$\begin{aligned}\mathcal{L}_3 = \frac{1}{2\kappa^2}R - \frac{1}{2}\partial_m D\partial^m D - \frac{1}{3!}e^{-\sqrt{2}\kappa D}H_{mnp}H^{mnp} - \frac{1}{\sqrt{2}\kappa}\sum_p \frac{1}{p!}e^{\frac{5-p}{\sqrt{2}}\kappa D}F_{m_1\dots m_p}F^{m_1\dots m_p}\\ +\sqrt{2}\sum_{p+q=8}(-1)^{[\frac{p+1}{2}]}\star(B\wedge F_{(p)}\wedge F_{(q)}) - \sqrt{2}\sum_p \frac{1}{p!}F_{j_1\dots j_p}B_{mn}F^{j_1\dots j_p mn}\,.\end{aligned} \tag{5.10}$$

[3]To compare, for example, with the conventions of [36], one should set $\sqrt{2}\kappa \to 1$ in the formulæ of the present paper, and make the following substitutions in equation (1.19) of that reference: $\phi \to D/\sqrt{2}$ (i.e. $\sigma \to e^{D/4}$), $\psi \to i\sqrt{2}\chi$, $\lambda \to i\sqrt{2}\lambda$, with all the remaining fields unchanged.

For type IIB, we are using a formalism with an action, so that the self-duality of the five-form (at lowest-order in α') is imposed at the level of the equations of motion. The last term of (5.10) contains couplings which depend on the NS and RR potentials, rather than just on the field strengths. As is well-known, they can be reabsorbed in the kinetic term for the RR fields if one introduces the modified field strengths

$$\widehat{F}_{(p)} = F_{(p)} + 2\kappa\ (-1)^p\, C_{(p-3)} \wedge H\ ; \qquad p \geq 3\ . \tag{5.11}$$

5.2 The 4-point amplitude

Up to an irrelevant normalization, the 4-point open-string amplitude in the $u-s$ channel can be written as:

$$\mathcal{A}_4^{\rm op}(u,s) = K_{SS}(k_i,a_i)G(u,s)\ , \tag{5.12}$$

where

$$\begin{aligned} K_{SS}(k_i;a_i,\xi_i) := 8\alpha'^2\Big\{ & 2k_1\cdot k_3\, k_1\cdot a_4\, k_2\cdot a_3\, a_1\cdot a_2 + \frac{1}{2}k_1\cdot k_4\, k_2\cdot k_4\, a_1\cdot a_2\, a_3\cdot a_4 \\ & + ik_1\cdot k_3\Big[a_2\cdot a_3\,(\xi_1 \not{k}_3 \xi_4) - k_3\cdot a_2\,(\xi_1 \not{a}_3 \xi_4) + k_2\cdot a_3\,(\xi_1 \not{a}_2 \xi_4)\Big] \\ & - ik_1\cdot k_4\Big[k_1\cdot a_3\,(\xi_1 \not{a}_2 \xi_4) + \frac{1}{2}(\xi_1 \not{a}_3 \not{k}_3 \not{a}_2 \xi_4)\Big] \\ & + \frac{1}{3}k_1\cdot k_2\ (\xi_1\gamma_m\xi_4)(\xi_2\gamma^m\xi_3)\Big\} + \text{permutations}\,, \end{aligned} \tag{5.13}$$

and $G(u,s)$ is defined in appendix D. It is useful to note that

$$\begin{aligned} t_{abcdefgh}M_1^{ab}M_2^{cd}M_3^{ef}M_4^{gh} = & -2\,(\mathrm{tr}M_1M_2\mathrm{tr}M_3M_4 + \mathrm{tr}M_1M_3\mathrm{tr}M_2M_4 + \mathrm{tr}M_1M_4\mathrm{tr}M_2M_3) \\ & + 8\,(\mathrm{tr}M_1M_2M_3M_4 + \mathrm{tr}M_1M_3M_2M_4 + \mathrm{tr}M_1M_3M_4M_2)\,. \end{aligned} \tag{5.14}$$

In particular for $f_i^{mn} := 2k_i^{[m}a_i^{n]}$, we find

$$\begin{aligned} t_{mnpqrstu}f_1^{mn}f_2^{pq}f_3^{rs}f_4^{tu} = & -8k_1\cdot k_3\, k_1\cdot a_4\, k_2\cdot a_3\, a_1\cdot a_2 \\ & -2k_1\cdot k_4\, k_2\cdot k_4\, a_1\cdot a_2\, a_3\cdot a_4 + \text{permutations}\ , \end{aligned} \tag{5.15}$$

so that the purely bosonic part of (5.13) is equal to $-2\alpha'^2 t_8 f^4$. Using the relations (5.3), we write for the closed 4-point amplitude

$$\mathcal{A}_4^{cl} = Nf(s,t,u)K_{SS}\otimes\tilde{K}_{SS}\ , \tag{5.16}$$

where N is a normalization factor to be fixed, and

$$\begin{aligned} f(s,t,u) = & \sin(-\pi\alpha'\frac{t}{2})G(\frac{\alpha's}{2},\frac{\alpha't}{2})G(\frac{\alpha't}{2},\frac{\alpha'u}{2}) \\ & \sim -\frac{8\pi}{\alpha'^3 stu} - 2\pi\zeta(3) + \mathcal{O}(\alpha'^2)\,. \end{aligned} \tag{5.17}$$

Proof of factorization

Let us now come to the proof of equation (1.1) mentioned already in the introduction. It is easy to see, concentrating for simplicity on the kinematic part and dropping the operator $\mathcal{G}$, that the 4-point open-string amplitude is of the form

$$\mathcal{A}_4^{op} \sim K_{SS} \sim \left(\widehat{\mathcal{L}}_4^{op} + \text{permutations}\right) ,$$

where $\widehat{\mathcal{L}}_4^{op}$ is obtained from the 4-point SYM Lagrangian (4.6) by splitting the positions of the fields and taking the Fourier transform. On the other hand, we can write

$$K_{SS} = k_{SS} + \text{permutations} ,$$

for some $k_{SS}(k_i; a_i, \xi_i)$, so that the permutations act on the positions of the particles. By comparing the two expressions it is clear that $\widehat{\mathcal{L}}_4^{op}$ can be identified with k_{SS}. It follows that the closed-string amplitude is of the form

$$\begin{aligned}\mathcal{A}_4^{cl} &\sim \left(k_{SS} + \text{permutations}\right) \otimes \left(\tilde{k}_{SS} + \text{permutations}\right) \\ &= K_{SS} \otimes \tilde{k}_{SS} + \text{permutations} ,\end{aligned}$$

where the permutations in the second line above act 'diagonally', i.e. they act simultaneously on both the left an the right sectors. It follows that $K_{SS} \otimes \tilde{k}_{SS}$ is obtained from the closed 4-point Lagrangian by splitting the positions of the fields and taking the Fourier transform. Recalling the relation of $\tilde{k}_{SS}$ to the 4-point open Lagrangian, we finally arrive at

$$\mathcal{L}_4^{cl} = \mathcal{L}_4^{op} \otimes \tilde{\mathcal{L}}_4^{op} ,$$

where the action of $\otimes$ should be understood as taking all pairs of fields formed by one field in the left and one field in the right sector, and using the (Fourier transform of the) factorization formulæ (3.10) to convert each pair of open fields to a closed field.

5.3 The closed-string effective action

- NS-NS

As was observed in [21], we can set $f_{ab} \otimes \tilde{f}_{cd} = \frac{1}{\kappa}\widehat{R}_{abcd}$, where we have introduced a modified Riemann tensor

$$\widehat{R}_{mn}{}^{pq} := R_{mn}{}^{pq} + 2\kappa e^{-\frac{\kappa D}{\sqrt{2}}} \nabla_{[m} H_{n]}{}^{pq} - \sqrt{2}\kappa \delta_{[m}{}^{[p} \nabla_{n]} \nabla^{q]} D . \tag{5.18}$$

The modified Riemann tensor can be thought of as corresponding to a connection with torsion. At the linearized level it obeys the following identities

$$\begin{aligned}\nabla_{[m} \widehat{R}_{np]qr} &= \nabla_{[m|} \widehat{R}_{np|qr]} = 0 \\ \nabla^i \widehat{R}_{imnp} &= \nabla^i \widehat{R}_{mnip} = 0 \\ \widehat{R}_{[mnp]q} &= \frac{2\kappa}{3} \nabla_q H_{mnp} \\ \widehat{R}_{mnpq}(H) &= \widehat{R}_{pqmn}(-H) ,\end{aligned} \tag{5.19}$$

which will be used in the following.

The NS-NS part of the bosonic effective action comes from the terms of the form $f^4 \otimes \tilde{f}^4$ in the closed four-point amplitude. We thus find

$$\mathcal{L}_{NS-NS} = -\frac{(\alpha')^3}{2^7 4! \pi \kappa^2} f(s,t,u) t_8 t_8 \widehat{R}^4 \ . \tag{5.20}$$

- $(\partial F)^2 R^2$

The two-graviton/two-RR part of the effective action, comes from cross terms of the form $f\xi^2 \otimes \tilde{f}\tilde{\xi}^2$ in the closed four-point amplitude. We thus find

$$\mathcal{L}_{(\partial F)^2 R^2} = \frac{(\alpha')^3}{2\pi\kappa} f(s,t,u)(A_1 + \frac{1}{2}A_2 + \frac{1}{4}A_3) \ . \tag{5.21}$$

In the equation above we have defined

$$\begin{aligned} A_1 &:= \widehat{R}^i{}_n{}^j{}_{n'} \widehat{R}_{ipjp'} < \gamma^n \partial^p \not{F} \gamma^{(n'} \partial^{p')} \not{F}^{Tr} > \\ A_2 &:= \widehat{R}_{mn}{}^i{}_{n'} \widehat{R}_{pqip'} \Big(< \gamma^{mnp} \partial^q \not{F} \gamma^{(n'} \partial^{p')} \not{F}^{Tr} > + < \gamma^{mnp} \partial^q \not{F}^{Tr} \gamma^{(n'} \partial^{p')} \not{F} > \Big) \\ A_3 &:= \widehat{R}_{mnm'n'} \widehat{R}_{pqp'q'} < \gamma^{[mnp} \partial^{q]} \not{F} \gamma^{m'n'p'} \partial^{q'} \not{F}^{Tr} > \ , \end{aligned} \tag{5.22}$$

where the notation $< \cdots >$ denotes the trace in the spinor indices. In order to bring (5.21) to this form, one has to perform an integration by parts, making use of the Bianchi identities and of the equations of motion (5.19), which leads to the following relations:

$$\begin{aligned} A_1 &= -\widehat{R}^i{}_n{}^j{}_{n'} \widehat{R}_{ipjp'} < \not{F} \gamma^{(n'} \partial^{p')} \partial^p \not{F}^{Tr} \gamma^n > \\ A_2 &= \widehat{R}_{mn}{}^i{}_{n'} \widehat{R}_{pqip'} \Big(< \not{F}^{Tr} \gamma^{(n'} \partial^{p')} \partial^q \not{F} \gamma^{mnp} > + < \not{F} \gamma^{(n'} \partial^{p')} \partial^q \not{F}^{Tr} \gamma^{mnp} > \Big) \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} &\widehat{R}_{mnm'n'} \widehat{R}_{pqp'q'} \Big(< \gamma^{[mnp} \partial^{q]} \not{F} \gamma^{m'n'p'} \partial^{q'} \not{F}^{Tr} > + < \not{F} \gamma^{[m'n'p'} \partial^{q']} \partial^q \not{F}^{Tr} \gamma^{mnp} > \Big) \\ &\qquad = \widehat{R}_{mnn'}{}^i \widehat{R}_{pip'q'} \Big(3 < \gamma^{mnp} \partial^{[n'} \not{F} \gamma^{p'} \partial^{q']} \not{F}^{Tr} > + \frac{1}{2} < \gamma^{mnp} \partial^j \not{F} \gamma^{n'p'q'} \partial_j \not{F}^{Tr} > \Big) \ . \end{aligned} \tag{5.24}$$

In addition, in the linearized approximation around flat space we have $R_{mn}{}^{pq} \sim \partial_{[m} \partial^{[p} h_{n]}{}^{q]}$. Taking this into account, one can prove the following identity

$$\widehat{R}_{mnn'}{}^i \widehat{R}_{pip'q'} \Big(3 < \gamma^{mnp} \partial^{[n'} \not{F} \gamma^{p'} \partial^{q']} \not{F}^{Tr} > + \frac{1}{2} < \gamma^{mnp} \partial^j \not{F} \gamma^{n'p'q'} \partial_j \not{F}^{Tr} > \Big) = 2A_3 \ , \tag{5.25}$$

or, equivalently,

$$A_3 = \widehat{R}_{mnm'n'} \widehat{R}_{pqp'q'} < \not{F} \gamma^{[m'n'p'} \partial^{q']} \partial^q \not{F}^{Tr} \gamma^{mnp} > \ . \tag{5.26}$$

Putting all the pieces together, we arrive at (5.21).

We can now compare our results to the corresponding ones in [34]. Indeed, equation (2.13) of that reference exactly reproduces equation (5.21) of the present paper. Note that had the authors of [34]

made a different choice of picture changing insertions, they would have instead arrived at terms of the form $F\partial^2 FR^2$, as on the right-hand sides of equations (5.23, 5.26). In other words, these two equations can be proven on-shell in the linearized approximation, by virtue of the picture-changing independence. In appendix F we shall give a brute-force derivation of (5.26) in the case of the $(\partial F_{(1)})^2 R^2$ couplings.

• $(\partial F)^4$

The purely RR part of the effective action comes from tensoring two copies of the purely-fermionic part of the open-string amplitude. We have

$$\mathcal{L}_{(\partial F)^4} = -\frac{(\alpha')^3}{36\pi} f(s,t,u)(B_1 - 5B_2 + B_3 + 4B_4 - B_5) \ , \tag{5.27}$$

where we have defined

$$\begin{aligned}
B_1 &:=< \partial_m\partial_p \not{F}\gamma_q \partial^m\partial^p \not{F}^{Tr}\gamma_n \not{F}\gamma^q \not{F}^{Tr}\gamma^n > \\
B_2 &:=< \partial_m\partial_p \not{F}\gamma_q \not{F}^{Tr}\gamma_n \partial^m\partial^p \not{F}\gamma^q \not{F}^{Tr}\gamma^n > \\
B_3 &:=< \partial_m\partial_p \not{F}\gamma_q \not{F}^{Tr}\gamma_n \not{F}\gamma^q \partial^m\partial^p \not{F}^{Tr}\gamma^n > \\
B_4 &:=< \partial_m\partial_p \not{F}\gamma_q \not{F}^{Tr}\gamma_n >< \partial^m\partial^p \not{F}\gamma^q \not{F}^{Tr}\gamma^n > \\
B_5 &:=< \not{F}\gamma_q \not{F}^{Tr}\gamma_n >< \partial^m\partial^p \not{F}\gamma^q \partial_m\partial_p \not{F}^{Tr}\gamma^n > \ .
\end{aligned} \tag{5.28}$$

In order to bring (5.27) to this form, we have integrated by parts, taking the linearized Bianchi identities and equations of motion into account, to arrive at the following relations:

$$\begin{aligned}
< \partial_m\partial_p \not{F}\gamma_q \not{F}^{Tr}\gamma_n >< \partial^m \not{F}\gamma^q \partial^p \not{F}^{Tr}\gamma^n > &= -B_4 + \frac{1}{2}B_5 \\
< \partial_m \not{F}\gamma_q \partial_p \not{F}^{Tr}\gamma_n >< \partial^m \not{F}\gamma^q \partial^p \not{F}^{Tr}\gamma^n > &= B_4 \\
< \partial_m\partial_p \not{F}\gamma_q \partial^m \not{F}^{Tr}\gamma_n \partial^p \not{F}\gamma^q \not{F}^{Tr}\gamma^n > &= \frac{1}{2}(-B_1 - B_2 + B_3) \\
< \partial_m\partial_p \not{F}\gamma_q \partial^m \not{F}^{Tr}\gamma_n \not{F}\gamma^q \partial^p \not{F}^{Tr}\gamma^n > &= \frac{1}{2}(-B_1 + B_2 - B_3) \\
< \partial_m\partial_p \not{F}\gamma_q \not{F}^{Tr}\gamma_n \partial^m \not{F}\gamma^q \partial^p \not{F}^{Tr}\gamma^n > &= \frac{1}{2}(B_1 - B_2 - B_3) \\
< \partial_m \not{F}\gamma_q \partial_p \not{F}^{Tr}\gamma_n \partial^m \not{F}\gamma^q \partial^p \not{F}^{Tr}\gamma^n > &= B_2 \ .
\end{aligned} \tag{5.29}$$

• $\psi\partial\psi(\partial F)^2$

These terms come from tensoring a copy of the purely-fermionic part of the open-string amplitude, with a copy of the quadratic-fermion part. We thus find

$$\begin{aligned}
\mathcal{L}_{\psi\partial\psi(\partial F)^2} =& \frac{i(\alpha')^3}{6\pi} f(s,t,u)\Big\{(\psi_{nk}\gamma_i\partial_j\psi^k{}_p) < \gamma^i\partial^j \not{F}\gamma^{(n}\partial^{p)}\not{F}^{Tr} > + (\psi_{nk}\gamma^i\partial^j \not{F}\gamma^{(n}\partial^{p)}\not{F}^{Tr}\gamma_i\partial_j\psi^k{}_p) \\
&+\frac{1}{2}(\psi_{mn}\gamma_i\partial_j\psi_{pq}) < \gamma^i\partial^j \not{F}\gamma^{[mnp}\partial^{q]}\not{F}^{Tr} > + \frac{1}{2}(\psi_{mn}\gamma^i\partial^j \not{F}\gamma^{[mnp}\partial^{q]}\not{F}^{Tr}\gamma_i\partial_j\psi_{pq})\Big\} \ .
\end{aligned} \tag{5.30}$$

• $\psi\partial\psi R^2$

These terms come from tensoring a copy of the purely-bosonic part of the open-string amplitude with a copy of the quadratic-fermion part. We thus find

$$\mathcal{L}_{\psi\partial\psi R^2} = \frac{i(\alpha')^3}{64\pi} f(s,t,u) t_8^{a_1\ldots a_8} \Big\{ R_{a_1a_2n}{}^{i} R_{a_3a_4pi} (\psi_{a_5a_6}\gamma^n\partial^p\psi_{a_7a_8}) + \frac{1}{2} R_{a_1a_2mn} R_{a_3a_4pq} (\psi_{a_5a_6}\gamma^{mnp}\partial^q\psi_{a_7a_8}) \Big\} \,. \tag{5.31}$$

• $\partial^2\psi^4$

These terms come from tensoring a copy of the purely-bosonic part of the open-string amplitude, with a copy of purely-fermionic part, or from $f^2\xi^2 \otimes \tilde{f}^2\tilde{\xi}^2$. We thus find

$$\begin{aligned}\mathcal{L}_{\partial^2\psi^4} = &-\frac{(\alpha')^3}{16\pi} f(s,t,u) \times \\ &\Big\{ \frac{1}{4!} t_8^{a_1\ldots a_8} (\psi_{a_1a_2}\gamma_i\partial_j\psi_{a_3a_4})(\psi_{a_5a_6}\gamma^i\partial^j\psi_{a_7a_8}) - (\psi_{mn}\gamma^{(n'}\partial^{p')}\psi_{pq})(\psi_{n'}{}^{i}\gamma^{[mnp}\partial^{q]}\psi_{p'i}) \\ &- (\psi_n{}^{i}\gamma^{(n'}\partial^{p')}\psi_{pi})(\psi_{n'}{}^{j}\gamma^{(n}\partial^{p)}\psi_{p'j}) - \frac{1}{4} (\psi_{mn}\gamma^{[m'n'p'}\partial^{q']}\psi_{pq})(\psi_{m'n'}\gamma^{[mnp}\partial^{q]}\psi_{p'q'}) \Big\} \,.\end{aligned} \tag{5.32}$$

Pole-subtraction

All terms in the Lagrangian come with a prefactor $f(s,t,u)$ which encodes the complete α' dependence of the amplitude. In practice, one is interested in knowing the Lagrangian at some given order in the α' expansion. The first term in the expansion of f, as seen in (5.17), has a pole of the form $1/(stu)$. This must be subtracted from the Lagrangian, since by unitarity all poles in an N-point amplitude must come from 1-particle-reducible diagrams containing N'-point vertices, where $N' < N$. However, the part proportional to $1/(stu)$ may also contain finite terms which contribute to the 4-point Lagrangian. In the NS sector there is a simple derivative-counting argument to show that there are no finite parts [21]. On the other hand, in the RR sector this can no longer be the case, as we expect to find 4-point couplings coming from the shifts (5.11) in the 3-point Lagrangian.

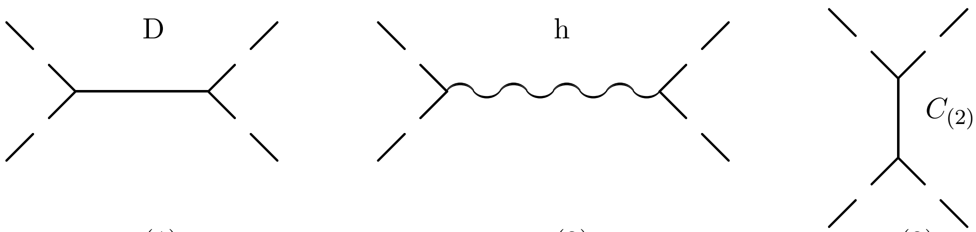
Let us illustrate the above discussion using as a concrete example the $C^2_{(0)}|H^2|$ term. The counting of derivatives shows that this coupling must come from the pole in f, since the finite piece contributes an amplitude $\sim F^2H^2$. The relevant part of the amplitude in (5.21), namely $H^2F^2_{(1)}$, yields:

$$-2\sqrt{2}\kappa\, \frac{t^2+u^2}{stu} k_3^i k_3^j\, (H_1)_{ilm}(H_2)_j{}^{lm} C_3^0 C_4^0 \,. \tag{5.33}$$

In order to arrive at this simple form, one must make use of the following relations which hold on-shell by virtue of the Bianchi identities:

$$\begin{aligned} k_3^i k_4^j (H_1)_{ilm}(H_2)_j{}^{lm} &= (\frac{u}{6}\delta^{ij} - k_3^i k_3^j)(H_1)_{ilm}(H_2)_j{}^{lm} \\ (k_3^i k_4^j - k_3^j k_4^i)(H_1)_{ilm}(H_2)_j{}^{lm} &= (\frac{u-t}{6}) H_1\cdot H_2 \,, \\ k_1^j k_2^i (H_1)_{ilm}(H_2)_j{}^{lm} &= -\frac{s}{6} H_1\cdot H_2 \,. \end{aligned} \tag{5.34}$$

There are three 1-particle-reducible diagrams which are relevant to this amplitude:



(1) (2) (3)

These diagrams contribute respectively:

$$\begin{aligned} F_1 =& c_1\, \delta^{ij}\,, \\ F_2 =& c_2\, \frac{tu}{stu} k_3^i k_3^j\,, \\ F_3 =& c_3\, \big(\frac{1}{3}\delta^{ij} + \frac{s^2}{stu} k_3^i k_3^j\big)\,. \end{aligned}$$

One can then check that the sum of the diagrams reproduces the amplitude (5.33) exactly up to the contact term generated by $|F_{(3)} - 2\kappa C_{(0)} H|^2$. The upshot is that the terms multiplying the singular part in the expansion of f are either generated by 1-particle-reducible diagrams, or are accounted for by the modification of the RR field strength. Only the regular part of f enters the quartic part of the Lagrangian, namely,

$$\widehat{\mathcal{G}}(s,t,u) := f(s,t,u) + \frac{8\pi}{\alpha'^3 stu}\,. \tag{5.35}$$

The Lagrangian

Collecting all the previous subsectors, taking the expansions (5.9) into account, we finally arrive at the complete four-point Lagrangian:

$$
\begin{aligned}
\mathcal{L}_4 = \frac{1}{2\kappa^2}R - \frac{1}{2}\partial_m D\partial^m D - \frac{1}{3!}e^{-\sqrt{2}\kappa D}H_{mnp}H^{mnp} - \frac{1}{\sqrt{2}\kappa}\sum_M \frac{1}{M!}e^{\frac{5-M}{\sqrt{2}}\kappa D}\widehat{F}_{m_1\dots m_M}\widehat{F}^{m_1\dots m_M}& \\
+ \sqrt{2}\sum_{M+N=8}(-1)^{[\frac{M+1}{2}]}\star(B\wedge F_{(M)}\wedge F_{(N)}) + \mathcal{O}(\psi^2)& \\
+ \quad \widehat{\mathcal{G}}(\partial,\alpha')\Big\{\frac{1}{4!}t^{a_1\dots a_8}t_{b_1\dots b_8}\widehat{R}_{a_1a_2}{}^{b_1b_2}\widehat{R}_{a_3a_4}{}^{b_3b_4}\widehat{R}_{a_5a_6}{}^{b_5b_6}\widehat{R}_{a_7a_8}{}^{b_7b_8}& \\
+ \sum_{M,N} u_{ij}{}^{mnpqm'n'p'q';a_1\dots a_M;b_1\dots b_N}\widehat{R}_{mnm'n'}\widehat{R}_{pqp'q'}\partial^i F_{a_1\dots a_M}\partial^j F_{b_1\dots b_N}& \\
+ \sum_{M,N,P,Q} v^{a_1\dots a_M;b_1\dots b_N;c_1\dots c_P;d_1\dots d_Q}\partial_i\partial_j F_{a_1\dots a_M}\partial^i\partial^j F_{b_1\dots b_N}F_{c_1\dots c_P}F_{d_1\dots d_Q}& \\
- 64i\sum_{M,N}\frac{c_M c_N \varepsilon_N}{M!N!}\partial^j F_{a_1\dots a_M}\partial^{(n|}F_{b_1\dots b_N}(\psi_{nk}\gamma^i\gamma^{a_1\dots a_M}\gamma^{|p)}\gamma^{b_1\dots b_N}\gamma_i\partial_j\psi^k{}_p)& \\
- 32i\sum_{M,N}\frac{c_M c_N \varepsilon_N}{M!N!}\partial^j F_{a_1\dots a_M}\partial^{[m|}F_{b_1\dots b_N}(\psi_{mn}\gamma^i\gamma^{a_1\dots a_M}\gamma^{|npq]}\gamma^{b_1\dots b_N}\gamma_i\partial_j\psi_{pq})& \\
+ 2it_8^{a_1\dots a_8}R_{a_1a_2n}{}^{i}R_{a_3a_4pi}(\psi_{a_5a_6}\gamma^n\partial^p\psi_{a_7a_8}) + it_8^{a_1\dots a_8}R_{a_1a_2mn}R_{a_3a_4pq}(\psi_{a_5a_6}\gamma^{mnp}\partial^q\psi_{a_7a_8})& \\
+ \frac{1}{3}t_8^{a_1\dots a_8}(\psi_{a_1a_2}\gamma_i\partial_j\psi_{a_3a_4})(\psi_{a_5a_6}\gamma^i\partial^j\psi_{a_7a_8}) - 8(\bar\psi_{mn}\gamma^{(n'}\partial^{p')}\bar\psi_{pq})(\psi_{n'}{}^{i}\gamma^{[mnp}\partial^{q]}\psi_{p'i})& \\
- 8(\bar\psi_n{}^{i}\gamma^{(n'}\partial^{p')}\bar\psi_{pi})(\psi_{n'}{}^{j}\gamma^{(n}\partial^{p)}\psi_{p'j}) - 2(\bar\psi_{mn}\gamma^{[m'n'p'}\partial^{q']}\bar\psi_{pq})(\psi_{m'n'}\gamma^{[mnp}\partial^{q]}\psi_{p'q'})& \\
+ (\psi\longleftrightarrow\bar\psi)\Big\}\ ,&
\end{aligned}
\tag{5.36}
$$

where the tensors u, v, are defined in appendix E. The sums over $M,\dots Q$, run over even integers from zero to four for IIA supergravity, and over odd integers from one to five for IIB. The action of the operator $\widehat{\mathcal{G}}$, cf. (5.35), should be understood in the same way as the action of $\mathcal{G}$ in (4.6).

6. The linearized superfield

For type IIB at order $(\alpha')^3$ (i.e. eight derivatives, or, R^4) we can compare our result to the prediction of the 'linearized superfield' of [31]: at the linearized level, in ten dimensional IIB superspace, one can define the analogue of a chiral scalar superfield. This is sometimes called the linearized, or the analytic, scalar superfield. We shall denote it here by A; it obeys the constraints

$$
\begin{aligned}
\bar D_\alpha A &= 0 \\
D^4 A &= \bar D^4 \bar A\ .
\end{aligned}
\tag{6.1}
$$

These constraints restrict the θ-expansion of A to the physical fields of IIB supergravity. In particular, the $\theta=0$ component is a complex scalar, the θ^2 component is a complex three-form, while the θ^4 component contains both the Riemann tensor and $\partial F_{(5)}$, and no new fields appear at higher orders in the θ-expansion. If A is the fluctuation around the flat-space solution, we expect the action

$$
\int d^{10}x\int d^{16}\theta A^4 \tag{6.2}
$$

to capture the four-field part of the action of type IIB at order $(\alpha')^3$, at the linearized level. Indeed, note that the action above is supersymmetric, up to terms quintic in the fields, and up to terms that vanish by virtue of the lowest-order (in α') equations of motion. Moreover, note that the θ-integration results in eight derivatives (on the bosonic part of the action).

In [52] it was argued, taking into account the constraints coming from the $SL(2,\mathbb{Z})$ invariance of type IIB, that the complete, to all string loops, action at order $(\alpha')^3$ is of the form

$$S_{(\alpha')^3} = f(\tau,\bar{\tau})\Big\{t_8 t_8 R^4 + \dots\Big\} \ , \tag{6.3}$$

where τ is the axiodilaton, and the ellipses stand for the remaining terms in the superinvariant. Unfortunately, it is not possible to use the linearized superfield to go beyond the four-field approximation[4]. For example, the action

$$\int d^{10}x \int d^{16}\theta A^5 \ , \tag{6.4}$$

would mix with (6.2), due to the nonlinear terms in the θ-expansion of (the full-fledged, non-linear extension of) A. Such terms are explicitly discussed in [38]. In going to quintic, or higher, order of interactions these nonlinear effects can no longer be ignored. This was explicitly verified by the authors of [29], who observed that the linearized superfield cannot reproduce the R^2H^3 terms in the string-theory effective action.

The authors of [34] observed that their result for the $(\partial F_{(5)})^2R^2$ terms in the string-theory effective action, is compatible with the prediction of (6.2). Let us review the argument: it was found in [34] that the $(\partial F_{(5)})^2R^2$ terms coming from string theory, can be written (by an appropriate field redefinition which amounts to setting $\lambda = 16$ in formula (2.13) of that reference) in such a way that only the (00200), (20011), (40000) representations occur[5] in the tensor-product decomposition of R^2. On the other hand, from the point-of-view of the linearized superfield, these terms come from (taking into account the fact that Grassmann integration can be thought of as differentiation)

$$(D^4A)^2(D^4A)^2 \sim (\partial F_{(5)})^2R^2 \ .$$

It follows that only representations in $(00001)^{8\otimes_a} \cap (02000)^{2\otimes_s}$ can occur in the decomposition of R^2. However, one can see that

$$(00001)^{8\otimes_a} \cap (02000)^{2\otimes_s} = (00200) \oplus (20011) \oplus (40000) \ ,$$

in agreement with string theory.

Note that a similar representation-theoretic argument cannot be used to compare the $(\partial F_{(1)})^2R^2$, $(\partial F_{(3)})^2R^2$ terms in the string-theory effective action to the corresponding terms in the linear-superfield action. The reason is that the two-axion, two-graviton terms in (6.2) come from

$$A(D^8A)(D^4A)^2 \sim C_{(0)}\partial^4 C_{(0)}R^2 \ ,$$

and can only be compared to the string-theory result after partial integration. Similarly the two-threeform, two-graviton terms in (6.2) come from

$$(D^2A)(D^6A)(D^4A)^2 \sim F_{(3)}\partial^2 F_{(3)}R^2 \ .$$

[4] A chiral measure in on-shell IIB superspace does not exist [37, 38].

[5] We are using the Dynkin notation for the complexification D_5 of $SO(1,9)$. Hence, (00000) is a scalar, (10000) is a vector, (01000) is a two-form, etc.

7. Discussion

It would be of interest to try to lift the ten-dimensional type IIA R^4-correction to eleven dimensions[6]. Partial results on such an attempt were reported in [30] and more recently in [48, 49] (see also [50] which addresses some problems with the computation of [48]). At present, in the absence of a superPoincaré-invariant microscopic formulation of M-theory, a covariant computation directly in eleven dimensions would have to rely on supersymmetry[7]. The current status of the superspace approach to higher-order derivative corrections in eleven dimensions can be found in [39], in which the supertorsion Bianchi identities in eleven dimensions are solved in all generality to first order in a deformation parameter (see [43] for earlier partial results). In [39] the deformations to the supertorsion constraints were parameterized in terms of certain superfields which were treated as 'black boxes'. In order to obtain explicit expressions however, these superfields would ultimately have to be expressed in terms of the physical fields in the massless multiplet. Unfortunately at present this remains a very difficult problem, equivalent to the computation of certain spinorial-cohomology groups, although a systematic way to arrive at these explicit corrections has been proposed in [44].

An obvious extension of the results in this paper is the investigation of higher-point, eight-derivative couplings. With the exception of the anomaly-related Chern-Simons terms in ten or eleven dimensions [45] (which appear at five points), this is a subject about which very little is known (see [29] for some partial results, and [34] for a general discussion). It would be of interest to examine whether the factorization property (1.1) can be generalized in any useful way to the case of quintic, or higher, interactions.

An important application of higher-order corrections, one which has recently attracted renewed interest, is the modification of the macroscopic properties of black-holes. String theory provides an underlying microscopic formulation within which the thermodynamic properties of black holes (at least of certain kinds thereof) can be derived with remarkable accuracy. It has been observed that higher-derivative (R^2 in four spacetime dimensions) contributions may lead to qualitatively different behavior, for example the appearance of a horizon even in the case where some of the black-hole charges vanish. It would be of interest to investigate the implications for this subject, of the higher-derivative corrections derived in the present paper.

In principle our result can be used to compute the corrections to the supersymmetry transformations to all orders in α', at the quartic approximation in the fields. This is of interest to the investigation of higher-derivative corrections to supersymmetric backgrounds. In particular, it would be desirable to discover contexts in which these corrections can qualitatively modify the geometrical properties of the background, e.g. smooth-out singularities, etc. We hope to report on this in the future.

[6] Spinorial-cohomology techniques [51] can be employed to show that the first higher-order correction in eleven dimensions appears at order l^3_{Planck} [40] (five derivatives). This correction is of topological nature and is related to the shifted quantization condition of the four-form field strength in eleven dimensions [41].

[7] Some results on this have appeared recently in [42]. These authors use the Nöther method to partially cancel the supersymmetric variation of a certain subsector of the action at the eight-derivative order.

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A. Fierz identities

For anticommuting spinors $\theta^\alpha, \phi^\beta$,

$$\theta^\alpha\phi^\beta = \frac{1}{16}(\gamma_m)^{\alpha\beta}(\theta\gamma^m\phi) + \frac{1}{96}(\gamma_{mnp})^{\alpha\beta}(\theta\gamma^{mnp}\phi) + \frac{1}{3840}(\gamma_{mnpqr})^{\alpha\beta}(\theta\gamma^{mnpqr}\phi), \tag{A.1}$$

from which it follows that

$$(\theta M\chi)(\theta N\psi) = -\frac{1}{96}(\theta\gamma^{mnp}\theta)(\chi M^{tr}\gamma_{mnp}N\psi) \ . \tag{A.2}$$

Using the above we can prove the following Fierz identities

$$\begin{aligned}(\gamma^{[i}{}_{mn}\theta)_\alpha(\theta\gamma^{j]mn}\theta) &= 2(\gamma_k\theta)_\alpha(\theta\gamma^{ijk}\theta)\\ (\gamma^{ijmnp}\theta)_\alpha(\theta\gamma_{mnp}\theta) &= -6(\gamma_k\theta)_\alpha(\theta\gamma^{ijk}\theta) \ .\end{aligned} \tag{A.3}$$

B. Zero-mode formulæ

The following is list a of formulæ repeatedly used in the derivation of the amplitudes:

$$T^{\alpha\beta\gamma}(\gamma^i\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{i'j'k'}\theta)F_{ijk,i'j'k'} = \frac{1}{120}F_{ijk,}{}^{ijk} \ , \tag{B.1}$$

where $F_{ijk,i'j'k'}$ is any tensor antisymmetric in the first three and in the last three indices and

$$T^{\alpha\beta\gamma} := T^{\alpha\beta\gamma}_{\rho\sigma\tau}(\frac{\partial}{\partial\theta}\gamma^{pmn}\frac{\partial}{\partial\theta})(\gamma_p\frac{\partial}{\partial\theta})^\rho(\gamma_m\frac{\partial}{\partial\theta})^\sigma(\gamma_n\frac{\partial}{\partial\theta})^\tau \ . \tag{B.2}$$

$$T^{\alpha\beta\gamma}(\gamma^{imn}\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{i'j'k'}\theta)F_{imn,jk,i'j'k'} = \frac{1}{70}F_{imn,}{}^i{}_j{}^{jmn} \ , \tag{B.3}$$

where $F_{imn,jk,i'j'k'}$ is any tensor antisymmetric in (i,m,n), in (j,k) and (i',j',k').

$$\begin{aligned}&T^{\alpha\beta\gamma}(\gamma^{m_1\dots m_5}\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{i'j'k'}\theta)F_{m_1\dots m_5,jk,i'j'k'}\\ &\qquad = -\frac{1}{42}\Big\{F^{m_1\dots m_5}{}_{,m_1m_2,m_3m_4m_5} + \frac{1}{5!}\varepsilon^{m_1\dots m_{10}}F_{m_1\dots m_5,m_6m_7,m_8m_9m_{10}}\Big\} \ ,\end{aligned} \tag{B.4}$$

where $F_{m_1\dots m_5,jk,i'j'k'}$ is any tensor antisymmetric in $(m_1\dots m_5)$, in (j,k) and (i',j',k'). Similarly, for an antichiral spinor $\bar\theta_\alpha$ we have:

$$\begin{aligned}&T_{\alpha\beta\gamma}(\gamma^{m_1\dots m_5}\bar\theta)^\alpha(\gamma^j\bar\theta)^\beta(\gamma^k\bar\theta)^\gamma(\bar\theta\gamma^{i'j'k'}\bar\theta)F_{m_1\dots m_5,jk,i'j'k'}\\ &\qquad = -\frac{1}{42}\Big\{F^{m_1\dots m_5}{}_{,m_1m_2,m_3m_4m_5} - \frac{1}{5!}\varepsilon^{m_1\dots m_{10}}F_{m_1\dots m_5,m_6m_7,m_8m_9m_{10}}\Big\} \ .\end{aligned} \tag{B.5}$$

We can prove (B.1) as follows. First note that the left-hand side is a scalar. On the other hand there is only one scalar in the tensor product of two three-forms

$$F_{ijk,i'j'k'} \sim (00100)^{2\otimes} = 1(00000) \oplus \dots$$

and we can take the right-hand side to be proportional to $F_{ijk,}{}^{ijk}$. The proportionality constant is determined by taking $F_{ijk,}{}^{i'j'k'} = \delta^{i'}_{[i}\delta^{j'}_{j}\delta^{k'}_{k]}$ and noting that $\delta^{i}_{[i}\delta^{j}_{j}\delta^{k}_{k]} = 120$.

Similarly, we can prove (B.3) by noting that there is only one scalar in the decomposition of the tensor product of two three-forms and a two-form:

$$F_{imn,jk,i'j'k'} \sim (00100)^{2\otimes} \otimes (01000) = 1(00000) \oplus \dots$$

and we can take the right-hand side to be proportional to $F_{imn,}{}^{i}{}_{j,}{}^{jmn}$. In order to determine the proportionality constant we set

$$F_{imn,}{}^{j}{}_{k,}{}^{i'j'k'} = \frac{1}{2}\Big(\delta^{[i'}_{[m}\delta^{j'}_{n|}\delta^{k']}_{k}\delta^{j}_{|i]} - \delta^{[i'}_{[m}\delta^{j'}_{n|}\eta^{k']j}\eta_{|i]k}\Big) ,$$

so that

$$T^{\alpha\beta\gamma}(\gamma^{imn}\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{i'j'k'}\theta)F_{imn,jk,i'j'k'} = T^{\alpha\beta\gamma}(\gamma^{jmn}\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{kmn}\theta) .$$

We then arrive at (B.3) by taking (A.3) into account and noting that

$$\frac{1}{2}\Big(\delta^{[m}_{[m}\delta^{n}_{n|}\delta^{k]}_{k}\delta^{j}_{|j]} - \delta^{[m}_{[m}\delta^{n}_{n|}\eta^{k]j}\eta_{|j]k}\Big) = 140 .$$

A consequence of (B.1, B.3) is the following formula

$$\begin{aligned}T^{\alpha\beta\gamma}(\gamma^i\gamma^{mn}\theta)_\alpha(\gamma^j\theta)_\beta(\gamma^k\theta)_\gamma(\theta\gamma^{i'j'k'}\theta)F_{i,mn,jk,i'j'k'}&\\ = \frac{1}{210}F_{i,mn;}{}^{i}{}_{k,}{}^{kmn} - \frac{1}{105}F_{m,in,}{}^{i}{}_{k,}{}^{kmn} + \frac{1}{60}F^{m}{}_{,mi,jk,}{}^{ijk} ,&\end{aligned} \tag{B.6}$$

where $F_{i,mn,jk,i'j'k'}$ is any tensor antisymmetric in (m,n), in (j,k) and (i',j',k').

We can prove (B.4) by noting that there are two scalars in the decomposition of the tensor product of a five-form, a three-form and a two-form. Hence we can take the right-hand side to be proportional to

$$\alpha F^{m_1\dots m_5}{}_{,m_1m_2,m_3m_4m_5} + \beta\varepsilon^{m_1\dots m_{10}}F_{m_1\dots m_5,m_6m_7,m_8m_9m_{10}} .$$

In order to determine the constant α we set $F^{m_1\dots m_5}{}_{,m_1m_2,m_3m_4m_5} = \delta^{m_1}_{[m_6}\dots\delta^{m_5}_{m_{10}]}$, taking (A.3) into account and noting that

$$\delta^{m_1}_{[m_1}\dots\delta^{m_5}_{m_5]} = 252 .$$

To determine β we set $F_{m_1\dots m_{10}} = \varepsilon_{m_1\dots m_{10}}$, taking the Hodge dualization

$$\gamma_{(n)} = (-)^{\frac{1}{2}n(n-1)} * \gamma_{(10-n)}\gamma_{11} \tag{B.7}$$

into account. Equation (B.5) is proven similarly.

C. Amplitudes

We can break the correlator (4.4) down according to the individual terms in V_4, as follows:

• $\partial\theta^\alpha A_\alpha$: This does not contribute.

• $\Pi^m A_m$: In this correlator, $\Pi^m \sim \partial x^m$. We can separate the contributions according to the number of θ's in the vertices.

- $\beta_1 : A_\alpha^{(1)} A_\beta^{(1)} A_\gamma^{(1)} A_m^{(2)}$

$$
\begin{aligned}
-\frac{\alpha'}{23040}\Pi(z_{ij}) \Big[& (\frac{a_1\cdot k_2}{z_2-z_4}+\frac{a_1\cdot k_3}{z_3-z_4})(a_2\cdot a_4\, a_3\cdot k_4 - a_2\cdot k_4\, a_3\cdot a_4)+ \\
& -(\frac{a_2\cdot k_1}{z_1-z_4}+\frac{a_2\cdot k_3}{z_3-z_4})(a_1\cdot a_4\, a_3\cdot k_4 - a_1\cdot k_4\, a_3\cdot a_4)+ \\
& +(\frac{a_3\cdot k_1}{z_1-z_4}+\frac{a_3\cdot k_2}{z_2-z_4})(a_1\cdot a_4\, a_2\cdot k_4 - a_1\cdot k_4\, a_2\cdot a_4)\Big] \,.
\end{aligned} \tag{C.1}
$$

- $\beta_2 : A_\alpha^{(2)} A_\beta^{(1)} A_\gamma^{(1)} A_m^{(1)}$

$$
\begin{aligned}
-\frac{i\alpha'}{34560}\frac{\Pi(z_{ij})}{z_1-z_4}\Big\{ & -\xi_2[\not{k}_1,\not{a}_1]\not{a}_3\xi_4 + 2a_1\cdot a_3\ \xi_2\not{k}_1\xi_4 + \xi_3[\not{k}_1,\not{a}_1]\not{a}_2\xi_4 \\
& -2a_1\cdot a_2\ \xi_3\not{k}_1\xi_4 + 4k_1\cdot a_2(\xi_3\not{a}_1\xi_4 - \xi_1\not{a}_3\xi_4) \\
& +4k_1\cdot a_3(\xi_1\not{a}_2\xi_4 - \xi_2\not{a}_1\xi_4)\Big\} - (2,1,3) - (3,2,1) \,.
\end{aligned} \tag{C.2}
$$

- $\beta_3 : A_\alpha^{(3)} A_\beta^{(1)} A_\gamma^{(1)} A_m^{(0)}$

$$
\begin{aligned}
-\frac{\alpha'}{5760}\Pi(z_{ij})\sum_r^3 \frac{k_r\cdot a_4}{z_r-z_4} & [a_1\cdot a_2\,(k_1-k_2)\cdot a_3 \\
& + a_1\cdot a_3\,(k_3-k_1)\cdot a_2 + a_2\cdot a_3\,(k_2-k_3)\cdot a_1]
\end{aligned} \tag{C.3}
$$

- $\beta_4 : A_\alpha^{(2)} A_\beta^{(2)} A_\gamma^{(1)} A_m^{(0)}$

$$
\frac{i\alpha'}{2880}\Pi(z_{ij})\sum_r^3 \frac{k_r\cdot a_4}{z_r-z_4}\,(\xi_1\not{a}_3\xi_2 - \xi_1\not{a}_2\xi_3 + \xi_2\not{a}_1\xi_3) \,. \tag{C.4}
$$

• $d_\alpha W^\alpha$: here d_α contributes with p_α and $\partial x^m(\theta\gamma_m)_\alpha$. The latter is easy to compute observing that, from the θ expansion, $\theta\gamma_m W^{(0)} = A_m^{(1)}$, and $\theta\gamma_m W^{(1)} = 2A_m^{(2)}$. Therefore these terms give $-\beta_1 - \frac{1}{2}\beta_2$.

The terms with p are:

- $\gamma_1 \,:\, U^{(1)}U^{(1)}U^{(1)}W^{(3)}$

  The correlator is equal to:

$$
\begin{aligned}
-\frac{\alpha'}{24\times 640}\Pi(z_{ij})\Big\{&\frac{a_1\cdot k_4}{z_1-z_4}[a_2\cdot k_4\;a_3\cdot a_4-a_3\cdot k_4\;a_2\cdot a_4]\\
&-\frac{a_2\cdot k_4}{z_2-z_4}[a_1\cdot k_4\;a_3\cdot a_4-a_3\cdot k_4\;a_1\cdot a_4]\\
&-\frac{a_3\cdot k_4}{z_3-z_4}[a_2\cdot k_4\;a_1\cdot a_4-a_1\cdot k_4\;a_2\cdot a_4]\Big\}\;.
\end{aligned}\tag{C.5}
$$

- $\gamma_2 \,:\, U^{(2)}U^{(1)}U^{(1)}W^{(2)}$

$$
\begin{aligned}
\frac{i\alpha'}{144\times 240}\Pi(z_{ij})\Big\{&\frac{1}{z_1-z_4}\,\frac{23}{4}(k_4\cdot a_2\,\xi_1\not{a}_3\xi_4-k_4\cdot a_3\,\xi_1\not{a}_2\xi_4)\\
&-\frac{1}{z_2-z_4}\,(4k_4\cdot a_2\,\xi_1\not{a}_3\xi_4+k_4\cdot a_3\,\xi_1\not{a}_2\xi_4)\\
&+\frac{1}{z_3-z_4}\,(k_4\cdot a_2\,\xi_1\not{a}_3\xi_4+4k_4\cdot a_3\,\xi_1\not{a}_2\xi_4)\Big\}\\
&-(2,1,3)-(3,2,1)
\end{aligned}\tag{C.6}
$$

- $\gamma_3 \,:\, U^{(3)}U^{(1)}U^{(1)}W^{(1)}$

  The correlator is equal to:

$$
\begin{aligned}
\frac{\alpha'}{192\times 240}\Pi(z_{ij})\Big\{&\frac{17}{2}\frac{1}{z_1-z_4}[-a_1\cdot k_4\;a_2\cdot k_1\;a_3\cdot a_4+a_1\cdot k_4\;a_3\cdot k_1\;a_2\cdot a_4\\
&\qquad-a_2\cdot k_4\;a_3\cdot k_1\;a_1\cdot a_4+a_2\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_3\\
&\qquad-a_3\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_2+a_3\cdot k_4\;a_2\cdot k_1\;a_1\cdot a_4\\
&\qquad-k_1\cdot k_4\;a_1\cdot a_3\;a_2\cdot a_4+k_1\cdot k_4\;a_1\cdot a_2\;a_3\cdot a_4]\\
&+\frac{1}{z_2-z_4}[-a_1\cdot k_4\;a_2\cdot k_1\;a_3\cdot a_4-4a_1\cdot k_4\;a_3\cdot k_1\;a_2\cdot a_4\\
&\qquad+a_1\cdot k_4\;a_4\cdot k_1\;a_2\cdot a_3+4a_2\cdot k_4\;a_3\cdot k_1\;a_1\cdot a_4\\
&\qquad-4a_2\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_3-a_3\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_2\\
&\qquad+a_2\cdot k_1\;a_3\cdot k_4\;a_1\cdot a_4-k_1\cdot k_4\;a_1\cdot a_4\;a_2\cdot a_3\\
&\qquad+4k_1\cdot k_4\;a_1\cdot a_3\;a_2\cdot a_4+k_1\cdot k_4\;a_1\cdot a_2\;a_3\cdot a_4]\\
&-\frac{1}{z_3-z_4}[-a_1\cdot k_4\;a_3\cdot k_1\;a_2\cdot a_4-4a_1\cdot k_4\;a_2\cdot k_1\;a_3\cdot a_4\\
&\qquad+a_1\cdot k_4\;a_4\cdot k_1\;a_2\cdot a_3+4a_3\cdot k_4\;a_2\cdot k_1\;a_1\cdot a_4\\
&\qquad-4a_3\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_2-a_2\cdot k_4\;a_4\cdot k_1\;a_1\cdot a_3\\
&\qquad+a_3\cdot k_1\;a_2\cdot k_4\;a_1\cdot a_4-k_1\cdot k_4\;a_1\cdot a_4\;a_2\cdot a_3\\
&\qquad+4k_1\cdot k_4\;a_1\cdot a_2\;a_3\cdot a_4+k_1\cdot k_4\;a_1\cdot a_3\;a_2\cdot a_4]\Big\}\\
&-(2,1,3)-(3,2,1)\;.
\end{aligned}\tag{C.7}
$$

- γ_4 : $U^{(2)}U^{(2)}U^{(1)}W^{(1)}$

$$\begin{aligned}
\frac{i\alpha'}{128\times 27\times 15}\Pi(z_{ij})\Big\{&\frac{1}{z_1-z_4}\Big[2a_3\cdot k_4\ \xi_1\not{a}_4\xi_2-2a_3\cdot a_4\ \xi_1\not{k}_4\xi_2-4\xi_1[\not{k}_4,\not{a}_4]\not{a}_3\xi_2\Big]\\
&-\frac{1}{z_2-z_4}\Big[2a_3\cdot k_4\ \xi_2\not{a}_4\xi_1-2a_3\cdot a_4\ \xi_2\not{k}_4\xi_1-4\xi_2[\not{k}_4,\not{a}_4]\not{a}_3\xi_1\Big]\\
&+\frac{9}{z_3-z_4}\Big[a_3\cdot a_4\ \xi_1\not{k}_4\xi_2-a_3\cdot k_4\ \xi_1\not{a}_4\xi_2\Big]\Big\}-(1,3,2)-(3,2,1)\ . \qquad \text{(C.8)}
\end{aligned}$$

- γ_5 : $U^{(4)}U^{(1)}U^{(1)}W^{(0)}$

$$\begin{aligned}
\frac{i\alpha'}{144}\Pi(z_{ij})\Big\{&-\frac{1}{64}\frac{1}{z_1-z_4}(k_1\cdot a_3\,\xi_1\not{a}_2\xi_4-k_1\cdot a_2\,\xi_1\not{a}_3\xi_4)+\\
&+\frac{1}{240}\frac{1}{z_2-z_4}\,k_1\cdot a_3\,\xi_1\not{a}_2\xi_4-\frac{1}{240}\frac{1}{z_3-z_4}\,k_1\cdot a_2\,\xi_1\not{a}_3\xi_4\Big\}\\
&-(2,1,3)-(3,2,1)\,. \qquad \text{(C.9)}
\end{aligned}$$

- γ_6 : $U^{(3)}U^{(2)}U^{(1)}W^{(0)}$

$$\begin{aligned}
\frac{i\alpha'}{96}\Pi(z_{ij})\Big\{&\frac{1}{z_1-z_4}(\frac{1}{72}k_1\cdot a_3\xi_2\not{a}_1\xi_4-\frac{1}{72}a_1\cdot a_3\xi_2\not{k}_1\xi_4+\frac{1}{90}k_1^a a_1^b a_3^c\xi_2\gamma_{abc}\xi_4)+\\
&+\frac{1}{z_2-z_4}(-\frac{1}{60}k_1\cdot a_3\xi_2\not{a}_1\xi_4+\frac{1}{60}a_1\cdot a_3\xi_2\not{k}_1\xi_4-\frac{1}{1440}k_1^a a_1^b a_3^c\xi_2\gamma_{abc}\xi_4)\Big\}\\
&+\frac{1}{z_3-z_4}(\frac{1}{360}k_1\cdot a_3\xi_2\not{a}_1\xi_4-\frac{1}{360}a_1\cdot a_3\xi_2\not{k}_1\xi_4-\frac{1}{360}k_1^a a_1^b a_3^c\xi_2\gamma_{abc}\xi_4)\Big\}\\
&+\text{cyclic perms.} \qquad \text{(C.10)}
\end{aligned}$$

- γ_7 : $U^{(2)}U^{(2)}U^{(2)}W^{(0)}$

$$\begin{aligned}
-\frac{\alpha'}{2880}\Pi(z_{ij})\Big\{&\frac{1}{z_1-z_4}(\xi_1\gamma^a\xi_4)\,(\xi_2\gamma_a\xi_3)+\\
&-\frac{1}{z_2-z_4}(\xi_2\gamma^a\xi_4)\,(\xi_1\gamma_a\xi_3)+\\
&+\frac{1}{z_3-z_4}(\xi_3\gamma^a\xi_4)\,(\xi_1\gamma_a\xi_2)\Big\}\ . \qquad \text{(C.11)}
\end{aligned}$$

- $N_{mn}F^{nm}$: This case is similar to the βs, as the contractions involve only the ghosts. One has $\lambda^\alpha(x)N_{mn}(y)\sim\frac{\alpha'}{2(x-y)}(\lambda\gamma_{mn})^\alpha$.

- δ_1 : $A_\alpha^{(1)}A_\beta^{(1)}A_\gamma^{(1)}A_m^{(2)}$

$$\begin{aligned}
-\frac{\alpha'}{64\times 144}\Pi(z_{ij})\Big\{&\frac{1}{z_1-z_4}(k_4\cdot a_1\,k_4\cdot a_2\,a_3\cdot a_4-k_4\cdot a_1\,k_4\cdot a_3\,a_2\cdot a_4)\Big\}\\
&-(2,1,3)-(3,2,1)\ . \qquad \text{(C.12)}
\end{aligned}$$

- δ_2 : $A_\alpha^{(2)}A_\beta^{(1)}A_\gamma^{(1)}A_m^{(1)}$

$$\begin{aligned}
&\frac{i\alpha'}{34560}\Pi(z_{ij})\Big\{\frac{1}{z_1-z_4}\frac{1}{2}(k_4\cdot a_2\,\xi_1\not{a}_3\xi_4-k_4\cdot a_3\xi_1\not{a}_2\xi_4)+\\
&\qquad-\frac{6}{z_2-z_4}k_4\cdot a_2\,\xi_1\not{a}_3\xi_4+\frac{6}{z_3-z_4}k_4\cdot a_3\xi_1\not{a}_2\xi_4)\Big\}\\
&\qquad-(2,1,3)+(3,1,2)\ . \qquad\qquad (C.13)
\end{aligned}$$

- δ_3 : $A_\alpha^{(3)}A_\beta^{(1)}A_\gamma^{(1)}A_m^{(0)}$

$$\begin{aligned}
&-\frac{\alpha'}{128\times 360}\Pi(z_{ij})\Big\{\frac{1}{2}\frac{1}{z_1-z_4}(-k_4\cdot a_1k_1\cdot a_2a_3\cdot a_4+k_4\cdot a_1k_1\cdot a_3a_2\cdot a_4\\
&\qquad-k_4\cdot a_2k_1\cdot a_3a_1\cdot a_4+k_4\cdot a_2k_1\cdot a_4a_1\cdot a_3\\
&\qquad-k_4\cdot a_3k_1\cdot a_4a_1\cdot a_2+k_4\cdot a_3k_1\cdot a_2a_1\cdot a_4\\
&\qquad-k_1\cdot k_4a_1\cdot a_3a_2\cdot a_4+k_1\cdot k_4a_1\cdot a_2a_3\cdot a_4)\\
&\qquad+\frac{1}{z_2-z_4}(-k_4\cdot a_1k_1\cdot a_2a_3\cdot a_4+4k_4\cdot a_1k_1\cdot a_3a_2\cdot a_4\\
&\qquad+k_4\cdot a_1k_1\cdot a_4a_2\cdot a_3-4k_4\cdot a_2k_1\cdot a_3a_1\cdot a_4\\
&\qquad+4k_4\cdot a_2k_1\cdot a_4a_1\cdot a_3-k_4\cdot a_3k_1\cdot a_4a_1\cdot a_2\\
&\qquad+k_4\cdot a_3k_1\cdot a_2a_1\cdot a_4-k_4\cdot k_1a_2\cdot a_3a_1\cdot a_4\\
&\qquad-4k_4\cdot k_1a_2\cdot a_4a_1\cdot a_3+k_4\cdot k_1a_1\cdot a_2a_3\cdot a_4)\\
&\qquad+\frac{1}{z_3-z_4}(-4k_4\cdot a_1k_1\cdot a_2a_3\cdot a_4+k_4\cdot a_1k_1\cdot a_3a_2\cdot a_4\\
&\qquad-k_4\cdot a_1k_1\cdot a_4a_2\cdot a_3-k_4\cdot a_2k_1\cdot a_3a_1\cdot a_4\\
&\qquad+k_4\cdot a_2k_1\cdot a_4a_1\cdot a_3-4k_4\cdot a_3k_1\cdot a_4a_1\cdot a_2\\
&\qquad+4k_4\cdot a_3k_1\cdot a_2a_1\cdot a_4+k_4\cdot k_1a_1\cdot a_4a_2\cdot a_3\\
&\qquad-k_4\cdot k_1a_1\cdot a_3a_2\cdot a_4+4k_4\cdot k_1a_1\cdot a_2a_3\cdot a_4)\Big\} \qquad\qquad (C.14)\\
&\qquad-(2,1,3)+(3,1,2)\ . \qquad\qquad (C.15)
\end{aligned}$$

- δ_4 : $A_\alpha^{(2)}A_\beta^{(2)}A_\gamma^{(1)}A_m^{(0)}$

$$\begin{aligned}
&\frac{i\alpha'}{36\times 96\times 15}\Big\{\frac{1}{z_1-z_4}(k_4^aa_3^ba_4^c\,\xi_1\gamma_{abc}\xi_2+a_3\cdot a_4\,\xi_1\not{k}_4\xi_2-a_3\cdot k_4\,\xi_1\not{a}_4\xi_2)+\\
&\qquad+\frac{1}{z_2-z_4}(-k_4^aa_3^ba_4^c\,\xi_1\gamma_{abc}\xi_2+a_3\cdot a_4\,\xi_1\not{k}_4\xi_2-a_3\cdot k_4\,\xi_1\not{a}_4\xi_2)+\\
&\qquad+\frac{1}{z_3-z_4}(9a_3\cdot a_4\,\xi_1\not{k}_4\xi_2-9a_3\cdot k_4\,\xi_1\not{a}_4\xi_2)\Big\}+\\
&\qquad-(1,3,2)+(2,3,1)\ . \qquad\qquad (C.16)
\end{aligned}$$

The full 4-boson amplitude comes from $\beta_3,\gamma_1,\gamma_3,\delta_1,\delta_3$. The result is

$$
\begin{aligned}
\frac{\alpha'}{5760}\Big[\frac{\Pi(z_{ij})}{z_1-z_4}\Big\{& 2\,k_1\cdot a_4\,k_2\cdot a_3\,a_1\cdot a_2 - 2\,k_1\cdot a_4\,k_3\cdot a_2\,a_1\cdot a_3 + \\
&+ 2k_3\cdot a_4\,k_2\cdot a_1\,a_2\cdot a_3 - 2k_2\cdot a_4\,k_3\cdot a_1\,a_2\cdot a_3 + \\
&+ 2k_1\cdot a_3\,k_3\cdot a_2\,a_1\cdot a_4 - 2k_1\cdot a_2\,k_2\cdot a_3\,a_1\cdot a_4 + \\
&- 2k_4\cdot a_1\,k_2\cdot a_3\,a_2\cdot a_4 + 2k_4\cdot a_1\,k_3\cdot a_2\,a_3\cdot a_4 + \\
&+ k_1\cdot k_4\,(a_1\cdot a_2\,a_3\cdot a_4 - a_1\cdot a_3\,a_2\cdot a_4) + (k_3-k_2)\cdot k_4\,a_2\cdot a_3\,a_1\cdot a_4\Big\}\Big] \\
&- (2,1,3) - (3,2,1)
\end{aligned}
\tag{C.17}
$$

The contribution from $\beta_2, \gamma_2, \delta_2, \gamma_5, \gamma_6$ is

$$
\begin{aligned}
\frac{i\alpha'}{24\times 5760}\Big[\frac{\Pi(z_{ij})}{z_1-z_4}\Big\{& \xi_1\not{a}_2\xi_4\,(2k_1\cdot a_3 + 49k_2\cdot a_3) + \xi_1\not{a}_3\xi_4\,(-2k_1\cdot a_2 - 49k_3\cdot a_2) \\
&+\xi_2\not{a}_1\xi_4\,(24k_1\cdot a_3 - 8k_2\cdot a_3) + \xi_3\not{a}_1\xi_4\,(-24k_1\cdot a_2 + 8k_3\cdot a_2) \\
&+\xi_2\not{a}_3\xi_4\,(-40k_2\cdot a_1 - 44k_3\cdot a_1) + \xi_3\not{a}_2\xi_4\,(44k_2\cdot a_1 + 40k_3\cdot a_1) \\
&-24a_1\cdot a_3\xi_2\not{k}_1\xi_4 + 24a_1\cdot a_2\xi_3\not{k}_1\xi_4 + 4a_1\cdot a_3\xi_2\not{k}_3\xi_4 - 4a_1\cdot a_2\xi_3\not{k}_2\xi_4 \\
&+24a_2\cdot a_3\xi_1\not{k}_3\xi_4 - 24a_2\cdot a_3\xi_1\not{k}_2\xi_4 + 2\xi_2[\not{k}_1,\not{a}_1]\not{a}_3\xi_4 - 2\xi_3[\not{k}_1,\not{a}_1]\not{a}_2\xi_4 \\
&+(k_2^a a_2^b a_3^c - k_3^a a_3^b a_2^c)\xi_1\gamma\xi_4 + (16k_1^a a_1^b a_3^c + 4k_3^a a_3^b a_1^c)\xi_2\gamma\xi_4 - (16k_1^a a_1^b a_2^c \\
&+4k_2^a a_2^b a_1^c)\xi_3\gamma\xi_4\Big\}\Big] - (2,1,3) - (3,2,1)
\end{aligned}
\tag{C.18}
$$

also equal to

$$
\begin{aligned}
\frac{i\alpha'}{5760}\Big[&\frac{\Pi(z_{ij})}{z_1-z_4}\Big\{2\xi_1\not{a}_2\xi_4\,k_2\cdot a_3 + 2\xi_2\not{a}_3\xi_4\,k_4\cdot a_1 + a_2\cdot a_3\xi_1\not{k}_3\xi_4 + \xi_3\not{a}_2\not{a}_1\not{k}_1\xi_4\Big\}\Big] \\
&+\text{cyclic}
\end{aligned}
\tag{C.19}
$$

The contribution from $\beta_4, \gamma_4, \delta_4$ is

$$
\begin{aligned}
\frac{i\alpha'}{5760}\Big[&\frac{\Pi(z_{ij})}{z_3-z_4}\Big\{\xi_1\not{a}_3\xi_2\,k_3\cdot a_4 - \xi_1\not{a}_4\xi_2\,k_4\cdot a_3 + 2\xi_2\not{a}_1\xi_3\,k_3\cdot a_4 + a_3\cdot a_4\xi_1\not{k}_4\xi_2 + \xi_1\not{a}_2\not{a}_4\not{k}_4\xi_3\Big\}\Big] \\
&+\text{cyclic}
\end{aligned}
\tag{C.20}
$$

D. Integrals

All integrals are reduced to hypergeometric functions using the formula

$$
\int_0^1 dt\,t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)\,,
$$

which is a single-valued analytic function of z with a cut on the real axis from 1 to ∞. We use the notations

$$y = \frac{z_2 - z_1}{z_3 - z_1}$$
$$p(y) = |z_2 - z_1|^c \, |z_3 - z_1|^b \, |z_3 - z_2|^a = \frac{y^c}{(1-y)^{b+c}} \; .$$

The positions of the vertices are ordered so that $z_1 < z_2 < z_3$, hence $0 < y < 1$. The formulae below are not valid outside this range. The integrals needed for the amplitudes are the following:

$$\begin{aligned}
& I_1(a,b,c;z_r) \equiv p(y) \int_{-\infty}^{\infty} \frac{dz_4}{z_1 - z_4} \, |z_1 - z_4|^a \, |z_2 - z_4|^b \, |z_3 - z_4|^c = \\
& = \left(c + a \, {}_2F_1(1,b;1-c;y)\right) \mathcal{G}(a,b,c) \\
& \quad + p(y)(-\Gamma(-c)\Gamma(1+c) - \psi(-c) + \psi(1+c)) \, ; \\
& I_2(a,b,c;z_r) \equiv p(y) \int_{-\infty}^{\infty} \frac{dz_4}{z_2 - z_4} \, |z_1 - z_4|^a \, |z_2 - z_4|^b \, |z_3 - z_4|^c = \\
& = a \, {}_2F_1(1,b;1-c;y) \mathcal{G}(a,b,c) \\
& \quad + p(y)(-\Gamma(-c)\Gamma(1+c) - \psi(-c) + \psi(1+c)) \, ; \\
& I_3(a,b,c;z_r) \equiv p(y) \int_{-\infty}^{\infty} \frac{dz_4}{z_3 - z_4} \, |z_1 - z_4|^a \, |z_2 - z_4|^b \, |z_3 - z_4|^c = \\
& = (a \, {}_2F_1(1,b;1-c;y) - a) \mathcal{G}(a,b,c) \\
& \quad + p(y)(-\Gamma(-c)\Gamma(1+c) - \psi(-c) + \psi(1+c)) \, ,
\end{aligned} \tag{D.1}$$

where

$$\begin{aligned}
\mathcal{G}(a,b,c) &:= \frac{\Gamma(a)\Gamma(b)}{\Gamma(1-c)} + \frac{\Gamma(a)\Gamma(c)}{\Gamma(1-b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(1-a)} \\
&:= G(a,b) + G(a,c) + G(b,c) \, .
\end{aligned} \tag{D.2}$$

On shell one has $a + b + c = 0$, and

$$\mathcal{G}(a,b,c) \sim -\frac{\pi^2}{2} + O(\alpha'^2) \; . \tag{D.3}$$

One can easily check that the combination $\sum_i \alpha_i I_i$ is y-independent and hence $SL(2,\mathbb{R})$-invariant iff $\sum \alpha_i = 0$. In fact,

$$I_1 - I_2 = c, \;\; I_1 - I_3 = -b, \;\; I_2 - I_3 = a \, . \tag{D.4}$$

E. Traces

The u, v coefficients in equations (1.2),(5.36), are defined as follows:

$$
\begin{aligned}
&u_{ij}{}^{mnpqm'n'p'q';a_1\dots a_M;b_1\dots b_N} := \\
&\qquad -32\frac{c_M c_N}{M!N!}\Big\{2g^{mq}g^{m'q'}\delta_i^p\delta_j^{(n'}\delta_k^{p')} < \gamma^n\gamma^{a_1\dots a_M}\gamma^k\gamma^{b_1\dots b_N} > \varepsilon_N \\
&\qquad\qquad -g^{m'q'}\delta_i^q\delta_j^{(n'}\delta_k^{p')} < \gamma^{mnp}\gamma^{a_1\dots a_M}\gamma^k\gamma^{b_1\dots b_N} > (\varepsilon_M+\varepsilon_N) \\
&\qquad\qquad +\frac{1}{2}\delta_i^{[q|}\delta_j^{q'} < \gamma^{|mnp]}\gamma^{a_1\dots a_M}\gamma^{m'n'p'}\gamma^{b_1\dots b_N} > \varepsilon_N\Big\}
\end{aligned}
\tag{E.1}
$$

and

$$
\begin{aligned}
&v^{a_1\dots a_M;b_1\dots b_N;c_1\dots c_P;d_1\dots d_Q} := \\
&\qquad \frac{32}{9}\frac{c_M c_N c_P c_Q}{M!N!P!Q!}\Big\{ < \gamma^{a_1\dots a_M}\gamma_q\gamma^{b_1\dots b_N}\gamma_n\gamma^{c_1\dots c_P}\gamma^q\gamma^{d_1\dots d_Q}\gamma^n > \varepsilon_N\varepsilon_Q \\
&\qquad\qquad - < \gamma^{a_1\dots a_M}\gamma_q\gamma^{b_1\dots b_N}\gamma_n >< \gamma^{c_1\dots c_P}\gamma^q\gamma^{d_1\dots d_Q}\gamma^n > \varepsilon_N\varepsilon_Q \\
&\qquad\qquad -5 < \gamma^{a_1\dots a_M}\gamma_q\gamma^{c_1\dots c_P}\gamma_n\gamma^{b_1\dots b_N}\gamma^q\gamma^{d_1\dots d_Q}\gamma^n > \varepsilon_P\varepsilon_Q \\
&\qquad\qquad +4 < \gamma^{a_1\dots a_M}\gamma_q\gamma^{c_1\dots c_P}\gamma_n >< \gamma^{b_1\dots b_N}\gamma^q\gamma^{d_1\dots d_Q}\gamma^n > \varepsilon_P\varepsilon_Q \\
&\qquad\qquad + < \gamma^{a_1\dots a_M}\gamma_q\gamma^{c_1\dots c_P}\gamma_n\gamma^{d_1\dots d_Q}\gamma^q\gamma^{b_1\dots b_N}\gamma^n > \varepsilon_P\varepsilon_N\Big\} ,
\end{aligned}
\tag{E.2}
$$

where

$$
\varepsilon_M := (-1)^{\frac{1}{2}M(M-1)} , \tag{E.3}
$$

and c_M was defined in (5.9). The explicit form of the traces above can readily obtained using a symbolic program for the manipulation of $\gamma\acute{\alpha}\mu\mu\alpha$ - matrices, e.g. [47][8]. Note that the result thus obtained will not necessarily be expressed in a basis of independent invariants; additional manipulations are needed if one wishes to bring the result to a form involving a minimal number of terms.

F. $(\partial F_{(1)})^2 R^2$

To illustrate the procedure, let us examine the $(\partial F_{(1)})^2 R^2$ couplings in more detail. First note that in the linearized approximation the equation of motion for $F_{(1)}$ reads $\partial^m F_m = 0$. In addition, $F_{(1)}$ must be closed by the Bianchi identities. These two conditions are equivalent to the statement that $\partial_m F_n$ is a traceless symmetric tensor. In the Dynkin notation for D_5:

$$
\partial_m F_n \sim (20000) .
$$

Similarly, at the linearized level, the equation of motion for the graviton reads $R_{mn} = 0$. In addition, the Riemann tensor obeys the Bianchi identities $R_{[mnp]q} = 0$. Together with the symmetry of the Riemann tensor $R_{mnpq} = R_{pqmn}$, these constraints can be expressed compactly as

$$
R_{mnpq} \sim (02000) .
$$

[8] In using [47], care should be taken to include the contribution of the totally-antisymmetric epsilon tensor in ten dimensions, which is not automatically taken care of by the program.

It follows that in the case at hand there are exactly five inequivalent scalars which can be constructed. In Dynkin notation:

$$(\partial F_{(1)})^2 R^2 \sim (20000)^{2\otimes_s} \otimes (02000)^{2\otimes_s} \sim 5 \times (00000) \oplus \dots \ .$$

Explicitly, we can choose a basis $I_1, \dots I_5$ of these five scalars as follows

$$\begin{aligned}
I_1 &:= \partial_m F^n \partial_p F^q R^{imjp} R_{injq} \\
I_2 &:= \partial_m F_n \partial^p F^q R^{imjn} R_{ipjq} \\
I_3 &:= \partial_m F^n \partial_p F^q R^{mpij} R_{ijnq} \\
I_4 &:= \partial_m F_i \partial^i F^n R^{mjkl} R_{njkl} \\
I_5 &:= \partial_i F_j \partial^i F^j R^{klmn} R_{klmn} \ .
\end{aligned} \tag{F.1}$$

In the linearized approximation around flat space we have in addition: $R_{mn}{}^{pq} \sim \partial_{[m}\partial^{[p}h_{n]}{}^{q]}$. Taking this into account, it is straightforward to show that in this approximation the invariants above are not independent, but obey

$$I_1 - I_2 + \frac{1}{2} I_3 + I_4 - \frac{1}{8} I_5 = 0 \ . \tag{F.2}$$

As we have argued in section 5.3, in the linearized approximation around flat space there is a relation

$$R_{mnm'n'} R_{pqp'q'} < \gamma^{[mnp}\partial^{q]} F \gamma^{m'n'p'} \partial^{q'} F^{Tr} >= R_{mnm'n'} R_{pqp'q'} < F \gamma^{[m'n'p'}\partial^{q']} \partial^q F^{Tr} \gamma^{mnp} > \ . \tag{F.3}$$

This can be explicitly verified in the case at hand: a straightforward computation yields

$$\begin{aligned}
R_{mnm'n'} R_{pqp'q'} < \gamma^{[mnp}\partial^{q]} F \gamma^{m'n'p'} \partial^{q'} F^{Tr} > &= 64(I_1 - I_2 + \frac{1}{2} I_3 + \frac{1}{2} I_4) \\
R_{mnm'n'} R_{pqp'q'} < F \gamma^{[m'n'p'}\partial^{q']} \partial^q F^{Tr} \gamma^{mnp} > &= -64(I_1 - I_2 + \frac{1}{2} I_3 + \frac{3}{2} I_4 - \frac{1}{4} I_5) \ .
\end{aligned} \tag{F.4}$$

The expressions on the right-hand sides of the two equations above can indeed be seen to be equal, when (F.2) is taken into account.

The couplings $(\partial F_{(1)})^2 R^2$ are related to the $(\partial D)^2 R^2$ couplings, coming from $t_8 t_8 \widehat{R}^4$, by $SL(2, \mathbb{Z})$ duality. We have directly verified that the sum of the two contributions is indeed $SL(2, \mathbb{Z})$ invariant, as expected.

References

[1] N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004** (2000) 018, hep-th/0001035.

[2] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," hep-th/0209059.

[3] B. E. W. Nilsson, "Off-shell fields for the 10-dimensional supersymmetric Yang-Mills theory", Göteborg-ITP-81-6; "Pure spinors as auxiliary fields in the ten-dimensional supersymmetric Yang-Mills theory," Class. Quant. Grav. **3** (1986) L41.

[4] P. S. Howe, "Pure spinors lines in superspace and ten-dimensional supersymmetric theories," Phys. Lett. B **258**, 141 (1991) [Addendum-ibid. B **259**, 511 (1991)].

[5] P. S. Howe, “Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions,” Phys. Lett. B **273** (1991) 90.

[6] N. Berkovits and B. C. Vallilo, “Consistency of super-Poincare covariant superstring tree amplitudes,” JHEP **0007** (2000) 015, hep-th/0004171.

[7] N. Berkovits, “Relating the RNS and pure spinor formalisms for the superstring,” JHEP **0108** (2001) 026, hep-th/0104247.

[8] N. Berkovits and O. Chandia, “Lorentz invariance of the pure spinor BRST cohomology for the superstring,” Phys. Lett. B **514** (2001) 394, hep-th/0105149.

[9] N. Berkovits, “Covariant multiloop superstring amplitudes,” hep-th/0410079.

[10] N. Berkovits, “Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring,” JHEP **0409** (2004) 047, hep-th/0406055.

[11] N. Berkovits and D. Z. Marchioro, “Relating the Green-Schwarz and pure spinor formalisms for the superstring,” JHEP **0501** (2005) 018, hep-th/0412198.

[12] N. Berkovits and C. R. Mafra, “Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms,” Phys. Rev. Lett. **96** (2006) 011602, hep-th/0509234.

[13] Y. Aisaka and Y. Kazama, “Origin of pure spinor superstring,” JHEP **0505** (2005) 046, hep-th/0502208.

[14] P. A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, “Covariant quantization of superstrings without pure spinor constraints,” JHEP **0210** (2002) 054, hep-th/0112162.

[15] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “The covariant quantum superstring and superparticle from their classical actions,” Phys. Lett. B **553** (2003) 96, hep-th/0209026.

[16] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “The quantum superstring as a WZNW model,” Nucl. Phys. B **676** (2004) 43, hep-th/0307056.

[17] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “The massless spectrum of covariant superstrings,” JHEP **0211** (2002) 001, hep-th/0202123.

[18] N. Berkovits, “Pure spinor formalism as an N = 2 topological string,” JHEP **0510** (2005) 089, hep-th/0509120.

[19] I. Oda and M. Tonin, “Y-formalism in pure spinor quantization of superstrings,” Nucl. Phys. B **727** (2005) 176, hep-th/0505277.

[20] A. Gaona and J. A. Garcia, “BFT embedding of the Green-Schwarz superstring and the pure spinor formalism,” JHEP **0509** (2005) 083 [arXiv:hep-th/0507076].

[21] D. J. Gross, J. H. Sloan, “The Quartic Effective action for the Heterotic String”, Nucl. Phys. B **291** (1987) 41.

[22] A. Kehagias and H. Partouche, “The exact quartic effective action for the type IIB superstring”, Phys. Lett. B **422** (1998) 109.

[23] A. Kehagias and H. Partouche, “D-instanton corrections as (p,q)-string effects and non-renormalization theorems,” Int. J. Mod. Phys. A **13** (1998) 5075.

[24] H. Kawai, D. C. Lewellen and S. H. H. Tye, “A Relation Between Tree Amplitudes Of Closed And Open Strings,” Nucl. Phys. B **269** (1986) 1.

[25] Z. Bern, “Perturbative quantum gravity and its relation to gauge theory,” Living Rev. Rel. **5** (2002) 5, gr-qc/0206071.

[26] P. A. Grassi and L. Tamassia, "Vertex operators for closed superstrings," JHEP **0407** (2004) 071, hep-th/0405072.

[27] I. N. McArthur, "Superspace Normal Coordinates," Class. Quant. Grav. **1** (1984) 233.

[28] D. Tsimpis, "Curved 11D supergeometry," JHEP **0411** (2004) 087, hep-th/0407244.

[29] K. Peeters, P. Vanhove and A. Westerberg, "Chiral splitting and world-sheet gravitinos in higher-derivative string amplitudes," Class. Quant. Grav. **19** (2002) 2699, hep-th/0112157.

[30] K. Peeters, P. Vanhove and A. Westerberg, "Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace," Class. Quant. Grav. **18** (2001) 843, hep-th/0010167.

[31] P. S. Howe and P. C. West, "The Complete N=2, D = 10 Supergravity," Nucl. Phys. B **238** (1984) 181.

[32] B. E. W. Nilsson and A. K. Tollsten, "Supersymmetrization of $\zeta(3)R^4_{\mu\nu\rho\sigma}$ in superstring theories," Phys. Lett. B **181** (1986) 63.

[33] M. B. Green, K. Peeters and C. Stahn, "Superfield integrals in high dimensions," JHEP **0508** (2005) 093, hep-th/0506161.

[34] K. Peeters and A. Westerberg, "The Ramond-Ramond sector of string theory beyond leading order," Class. Quant. Grav. **21** (2004) 1643, hep-th/0307298.

[35] J. J. Atick and A. Sen, "Covariant One Loop Fermion Emission Amplitudes In Closed String Theories," Nucl. Phys. B **293** (1987) 317.

[36] F. Giani and M. Pernici, "N=2 Supergravity In Ten-Dimensions," Phys. Rev. D **30** (1984) 325.

[37] N. Berkovits and P. S. Howe, unpublished.

[38] S. de Haro, A. Sinkovics and K. Skenderis, "On a supersymmetric completion of the R^4 term in IIB supergravity," Phys. Rev. D **67** (2003) 084010, hep-th/0210080.

[39] M. Cederwall, U. Gran, B. E. W. Nilsson and D. Tsimpis, "Supersymmetric corrections to eleven-dimensional supergravity," JHEP **0505**, 052 (2005), hep-th/0409107.

[40] D. Tsimpis, "11D supergravity at $\mathcal{O}(l^3)$," JHEP **0410**, 046 (2004), hep-th/0407271.

[41] E. Witten, "On flux quantization in M-theory and the effective action," J. Geom. Phys. **22** (1997) 1, hep-th/9609122.

[42] Y. Hyakutake and S. Ogushi, "Higher derivative corrections to eleven dimensional supergravity via local supersymmetry," JHEP **0602** (2006) 068, hep-th/0601092.

[43] M. Cederwall, U. Gran, M. Nielsen and B. E. W. Nilsson, "Manifestly supersymmetric M-theory," JHEP **0010** (2000) 041, hep-th/0007035; "Generalised 11-dimensional supergravity," hep-th/0010042.

[44] P. S. Howe and D. Tsimpis, "On higher-order corrections in M theory," JHEP **0309** (2003) 038, hep-th/0305129.

[45] M. J. Duff, J. T. Liu and R. Minasian, "Eleven-dimensional origin of string / string duality: A one-loop test," Nucl. Phys. B **452** (1995) 261, hep-th/9506126.

[46] M. de Roo, M. G. C. Eenink, "The effective action for the 4-point functions in abelian open superstring theory", hep-th/0307211.

[47] U. Gran, "GAMMA: A Mathematica package for performing Gamma-matrix algebra and Fierz transformations in arbitrary dimensions," hep-th/0105086.

[48] L. Anguelova, P. A. Grassi and P. Vanhove, "Covariant one-loop amplitudes in D = 11," Nucl. Phys. B **702** (2004) 269.

[49] P. A. Grassi and P. Vanhove, "Topological M theory from pure spinor formalism," hep-th/0411167.

[50] C. R. Mafra, "Four-point one-loop amplitude computation in the pure spinor formalism," JHEP **0601** (2006) 075.

[51] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, "Spinorial cohomology and maximally supersymmetric theories," JHEP **0202**, 009 (2002), hep-th/0110069; "The structure of maximally supersymmetric Yang-Mills theory: Constraining higher-order corrections," JHEP **0106**, 034 (2001), hep-th/0102009.

[52] M. B. Green and M. Gutperle, "Effects of D-instantons," Nucl. Phys. B **498** (1997) 195, hep-th/9701093.